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MATHEMATICS

On Some Properties of Convex Functions

by

TSUYOSHI ANDÔ

Presented by W. ORLICZ on April 26, 1960

M. A. Krasnoselskiĭ and Y. B. Rutickiĭ ([1], chap. 1) classified many properties of convex functions in connection with the theory of Orlicz spaces. However, as they state ([1], pp. 259—260), some properties complementary to those already classified have not been determined. In this paper a complete classification will be given.

1. A positive *convex* function $M(\xi)$ on $(0, \infty)$ is called an *N-function*, if $\frac{M(\xi)}{\xi} \to 0$ and $\frac{M(\xi)}{\xi} \to \infty$, for $\xi \to 0$ and $\xi \to \infty$, respectively. It is well known (see [1], par. 1) that $M(\xi)$ admits a non-negative, non-decreasing right derivative $p(\xi)$. The N-function $N(\xi)$, *complementary* to the given $M(\xi)$, is defined by

$$N(\xi) = \int_0^{\xi} q(\zeta)\, d\zeta,$$

where $q(\xi)$ is the right inverse of $p(\xi)$. It is sometimes difficult to obtain an explicit form of the complementary function. On the other hand, in order to investigate topological properties of Orlicz spaces (constructed on the bounded set of a Euclidean space) exact forms of N-functions are not necessary. On the contrary, the notion of equivalence is useful here, namely two N-functions $M_1(\xi)$ and $M_2(\xi)$ are said to be *equivalent* if there exist $\alpha, \beta > 0$ such that

$$M_1(\alpha\xi) \leqslant M_2(\xi) \leqslant M_1(\beta\xi) \quad \text{for large } \xi.$$

Orlicz spaces with respect to mutually equivalent N-functions consist of the same functions (see [1], par. 9]). In this paper all properties considered are invariant under equivalence.

We shall use only elementary properties of N-functions:

(1) $\frac{M(\xi)}{\xi}$ is increasing,

(2) $$\frac{1}{2}\,\xi \cdot p\left(\frac{1}{2}\,\xi\right) \leqslant M(\xi) \leqslant \xi \cdot p(\xi) \text{ for all } \xi.$$

The first property follows from the definition, and the latter from the fact that $p(\xi)$ is non-decreasing and $M(\xi)=\int_0^{\xi} p(\zeta)\,d\zeta$.

Each N-function $M(\xi)$ is equivalent to another $\Phi(\xi)$ which is differentiable and strictly convex.

In fact, on account of (1) and (2), we may take

$$\Phi(\xi)=\int_0^{\xi}\frac{M(\zeta)}{\zeta}\,d\zeta.$$

On the basis of this fact we shall assume in this paper that all N-functions are differentiable and strictly convex, that is, $p(\xi)$ is the derivative of $M(\xi)$ and $q(\xi)$ is the inverse of $p(\xi)$. $\Psi(\xi)$ denotes a general N-function. We define the following properties $(\triangle_2)$ and (∇_2), of $\Psi(\xi)$:

$$(\triangle_2)\qquad \Psi(2\xi)\leqslant K\Psi(\xi) \quad \text{for large } \xi,$$

$$(\nabla_2)\qquad 2K'\Psi(\xi)\leqslant\Psi(K'\xi) \quad \text{for large } \xi,$$

where $K, K'>1$ are some constants. In general, properties $\triangle$ and ∇ (concerning N-functions) are said to be mutually *complementary* in the case, when $\Psi(\xi)$ has the property $\triangle$ if and only if its complementary function has the property ∇. It is shown ([1], par. 4) that $(\triangle_2)$ and (∇_2) are mutually complementary. Moreover, on account of (2), we can prove that $M(\xi)$ has the property $(\triangle_2)$ if and only if its derivative $p(\xi)$ also has it.

Now we turn to multiplicative properties: Let the properties $(\triangle')$ and (∇') be defined thus:

$$(\triangle')\qquad \Psi(\xi\eta)\leqslant K\Psi(\xi)\,\Psi(\eta) \quad \text{for large } \xi,\ \eta,$$

$$(\nabla')\qquad \Psi(\xi)\cdot\Psi(\eta)\leqslant\Psi(K'\xi\eta) \quad \text{for large } \xi,\ \eta,$$

where $K, K'>0$ are some constants.

THEOREM 1. *$(\triangle')$ and (∇') are mutually complementary.*

Proof. Let $M(\xi)$ have the property $(\triangle')$. Then, on account of (2), its derivative $p(\xi)$ also has the property $(\triangle')$. Consequently, the inverse $q(\xi)$ has the property (∇'). Finally, using again (2), we can easily see that $N(\xi)$ itself has the property (∇'). The proof of the converse is similar.

Krasnoselskiĭ and Rutickiĭ ([1], par. 60) proved Theorem 1 under an additional condition $(\triangle_3)$ (for the definition see par. 2). They also conjectured that for each $M(\xi)$ with the property $(\triangle')$ there exists another $\Phi(\xi)$ which is equivalent to it and satisfies:

$$\Phi(\xi\eta)\leqslant K\cdot\Phi(\xi)\,\Phi(\eta) \quad \text{for all } \xi,\eta>0.$$

However, this assertion does not hold. In fact, although $M(\xi) = (1+\xi)\log(1+\xi) - \xi$ has property $(\triangle')$, it does not admit the desired equivalent function. Assume that there exists a $\Phi(\xi)$ such that

$$M(\alpha\xi) \leqslant \Phi(\xi) \leqslant M(\beta\xi) \quad \text{for } \xi \geqslant \xi_0,$$

for each fixed $0 < \xi < 1$. If $\eta \geqslant \frac{\xi_0}{\xi}$, we have

$$\frac{\Phi(\xi\eta)}{\Phi(\eta)} \geqslant \frac{M(\alpha\xi\eta)}{M(\beta\eta)} = \frac{(1+\alpha\xi\eta)\cdot\log(1+\alpha\xi\eta) - \alpha\xi\eta}{(1+\beta\eta)\cdot\log(1+\beta\eta) - \beta\eta}.$$

Consequently,

$$\lim \inf_{\eta\to\infty} \frac{\Phi(\xi\eta)}{\Phi(\eta)} \geqslant \frac{\alpha\xi}{\beta}.$$

On the other hand, since $(\triangle')$ is valid for all ξ, η we have

$$\lim \sup_{\eta\to\infty} \frac{\Phi(\xi\eta)}{\Phi(\eta)} \leqslant \lim_{\eta\to\infty} \frac{K\Phi(\xi)\,\Phi(\eta)}{\Phi(\eta)} = K\Phi(\xi).$$

Thus, we obtain $K\Phi(\xi) \geqslant \frac{\alpha\xi}{\beta}$ for all $\xi > 0$, which is in contradiction to $\frac{\Phi(\xi)}{\xi} \to 0$ for $\xi \to 0$.

2. One of the properties derived from $(\triangle_2)$ is that $\Psi(\xi)$ increases more slowly than some power, that is, $\Psi(\xi) \leqslant \xi^p$ for large ξ, for $p > 1$ constant. Similarly, if $\Psi(\xi)$ has the property $(\triangledown_2)$, it increases more rapidly than some power. This leads us to study conditions which imply that $\Psi(\xi)$ increases more slowly (resp. more rapidly) than all powers. For this purpose we introduce the following notions: $\Psi(\xi)$ has the property $(\triangle_\Phi)$, if

$$(\triangle_\Phi) \qquad \Phi[\Psi(\xi)] \leqslant \Psi(K\xi) \quad \text{for large } \xi,$$

where $\Phi(\xi)$ is an N-function and $K > 0$ is a constant. By simple calculation it is shown that $\Psi(\beta\xi) \geqslant a\Psi(\xi)$ implies $\Psi(\xi) \geqslant K\xi\log_\beta^\alpha$. Hence, if $\Psi(\xi)$ has the property $(\triangle_\Phi)$, it increases more rapidly than all powers, because $\frac{\Phi(\xi)}{\xi} \to \infty$ for $\xi \to \infty$.

Our next task will be to find the property complementary to $(\triangle_\Phi)$.

Let $\Phi[M(\xi)] \leqslant M(K\xi)$ for large ξ. Then, on account of (2),

$$\xi p(\xi)\cdot\varphi[\xi p(\xi)] \leqslant M(2\xi)\cdot\varphi[M(2\xi)] \leqslant$$

$$\leqslant \Phi[M(4\xi)] \leqslant M(4K\xi) \leqslant 4K\xi\cdot p(4K\xi),$$

where $\varphi(\xi)$ is the derivative of $\Phi(\xi)$. Writing $K' = 4K$ and $\eta = p(\xi)$, we have

$$\eta\cdot\varphi[\eta q(\eta)] \leqslant K'\cdot p[K' q(\eta)],$$

i.e. $$q\left[\frac{\eta\cdot\varphi[\eta q(\eta)]}{K'}\right]\leqslant K'\cdot q(\eta) \quad \text{for large } \eta,$$

since $p(\xi)$ and $q(\eta)$ are mutually inverse. Since $(\triangle_\Phi)$ implies (∇_2), $q(\eta)$, by Theorem 1, has the property $(\triangle_2)$. Consequently, we obtain

$$q[\eta\cdot\varphi[\eta\cdot q(\eta)]]\leqslant K''\,q(\eta),$$

where $K''>0$ is constant. Using (2) again, we obtain

$$q\left[\frac{\Phi[N(\eta)]}{q(\eta)}\right]\leqslant K''\,q(\eta).$$

Thus,

$$N\left[\frac{\eta\Phi[N(\eta)]}{N(\eta)}\right]\leqslant K''\,\Phi[N(\eta)] \quad \text{for large } \eta.$$

Writing $\xi=N(\eta)$, we have

$$N\left[\frac{N^{-1}(\xi)\Phi(\xi)}{\xi}\right]\leqslant K''\,\Phi(\xi),$$

finally

$$\frac{N^{-1}(\xi)\cdot\Phi(\xi)}{\xi}\leqslant K''\cdot N^{-1}[\Phi(\xi)] \quad \text{for large } \xi.$$

Since $\dfrac{N^{-1}(\xi)}{\xi}$ decreases to the limit 0 for $\xi\to\infty$, the property just obtained is better illustrated by the following inequality

$$\frac{N^{-1}(\xi)}{\xi}\leqslant\frac{K''\cdot N^{-1}[\Phi(\xi)]}{\Phi(\xi)} \quad \text{for large } \xi.$$

Now we introduce the following notion: $\Psi(\xi)$ has property (∇_Φ), if

$$(\nabla_\Phi)\qquad \frac{\Psi^{-1}(\xi)}{\xi}\leqslant\frac{K\cdot\Psi^{-1}[\Phi(\xi)]}{\Phi(\xi)} \quad \text{for large } \xi,$$

where $K>0$ is constant. Likewise, we can prove that, if $N(\xi)$ has property (∇_Φ) the complementary function $M(\xi)$ has the property $(\triangle_\Phi)$. Thus, we have proved

THEOREM 2. *$(\triangle_\Phi)$ and (∇_Φ) are mutually complementary.*

As shown above, property (∇_Φ) is a restriction on the degree of decrease of the function $\dfrac{\Psi^{-1}(\xi)}{\xi}$. It may also be considered as a restriction on the degree of increase of the function $\dfrac{\Psi(\xi)}{\xi}$. On the other hand, Krasnoselskiĭ and Rutickiĭ ([1], par 6) considered some conditions under which $M[\Phi(\xi)]$ is equivalent to $\dfrac{M(\xi)\cdot\Phi(\xi)}{\xi}$. But from our standpoint, it is

more convenient to express this relation as a restriction on the function $\frac{\Psi(\xi)}{\xi}$: $\Psi(\xi)$ has property (∇_{Φ}^{*}) if

$$(\nabla_{\Phi}^{*}) \qquad \frac{\Psi[\Phi(\xi)]}{\Phi(\xi)} \leqslant \frac{K\Psi(\xi)}{\xi} \quad \text{for large } \xi,$$

where $K > 0$ is constant. It is easily seen that (∇_{Φ}) implies (∇_{Φ}^{*}), and the latter in turn implies that $\Psi(\xi)$ increases more slowly than all powers of degrees $p > 1$. Analogously to Theorem 2, we can prove

THEOREM 3. *The property complementary to* (∇_{Φ}^{*}) *is the following*:

$$(\triangle_{\Phi}^{*}) \qquad \Phi\left[\frac{\Psi(\xi)}{\xi}\right] \leqslant \frac{\Psi(K\xi)}{\xi} \quad \textit{for large } \xi,$$

where $K > 0$ *is constant.*

As a property which implies more rapid increase of $M(\xi)$ than all powers, Krasnoselskiĭ and Rutickiĭ ([1], par 6) introduced the property:

$$(\triangle_3) \qquad M(\xi) \text{ is equivalent to } \xi \cdot M(\xi),$$

and proved that $M(\xi)$ has property $(\triangle_3)$ if and only if $N[M(\xi)]$ is equivalent to it (where $N(\xi)$ is complementary to it). In our notations $M(\xi)$ has the property $(\triangle_3)$ if and only if it has the property $(\triangle_N)$. Thus, by Theorem 2, the following property complementary to $(\triangle_3)$ is obtained:

$M(\xi)$ has property $(\triangle_3)$ if and only if the complementary function $N(\xi)$ has the following property:

$$(\nabla_3) \qquad N^{-1}(\xi) \cdot N(\xi) \leqslant K \cdot \xi^2 \quad \textit{for large } \xi,$$

where $K > 0$ *is constant.*

Since under the condition $(\triangle_3)$ $M(\xi)$ is equivalent to $\frac{M(\xi)}{\xi}$, we can affirm: *under the condition* $(\triangle_3)$ $M(\xi)$ *has the property* $(\triangle_{\Phi})$ *if and only if the complementary function* $N(\xi)$ *has the property* (∇_{Φ}^{*}).

If we put $\Phi(\xi) = \xi^2$ in the definition of $(\triangle_{\Phi})$, we obtain a property, which Krasnoselskiĭ and Rutickiĭ called $(\triangle^2)$. They obtained the property complementary to it as follows ([1], par. 6):

$$(\nabla^2) \qquad \frac{N(\xi)}{\xi} \leqslant \frac{K \cdot N(\sqrt{\xi})}{\sqrt{\xi}} \quad \text{for large } \xi,$$

where $K > 0$ is constant. Since $(\triangle^2)$ implies $(\triangle_3)$, this result can be derived immediately from Theorem 3.

In the definitions of (∇_{Φ}^{*}) and $(\triangle_{\Phi}^{*})$, $\frac{\Psi(\xi)}{\xi}$ may be replaced by $\Psi(\xi)$ (its derivative), since $\frac{\Psi(\xi)}{\xi}$ and $\Psi(\xi)$ are equivalent.

3. In concluding we mention implications among the properties. It is easy to see that $(\triangle^2)\to(\triangle_3)\to(\nabla_2)$ and $(\triangle^2)\to(\nabla')\to(\nabla_2)$, where $\to$ indicates implication. (∇') is incomparable with $(\triangle_3)$. In fact, $M(\xi)=\xi^p$ $(p>1)$ has the property (∇') but not $(\triangle_3)$. And putting

$$p(\xi)=\begin{cases}\xi & \text{for} \quad 0\leqslant\xi<2\\ (n\xi)^{2^{(n+1)^2}} & \text{for} \quad 2^{2^n}\leqslant\xi<2^{2^{n+1}}\end{cases}$$

and $M(\xi)=\int_0^\xi p(\zeta)\,d\zeta$, we can easily see, that $M(\xi)$ has $(\triangle_3)$ but not (∇'). Similarly, $(\nabla^2)\to(\nabla_3)\to(\triangle_2)$ and $(\nabla^2)\to(\triangle')\to(\triangle_2)$. There is no other implication among them.

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MATHEMATICS

On Certain Characterization of Weak Convergence of Measures

by

R. BARTOSZYŃSKI

Presented by E. MARCZEWSKI on May 2, 1960

Let R be a separable complete metric space with the metric ϱ, $\mathcal{B}$ — the class of all Borel subsets of R. We shall denote by $M(R)$ the space of all finite measures μ defined on $\mathcal{B}$. A sequence $\mu_n \in M(R)$ will be called weakly convergent to $\mu \in M(R)$, if for every bounded and continuous function $f(x)$ on R

$$\lim_{n\to\infty} \int_R f(x)\,\mu_n(dx) = \int_R f(x)\,\mu(dx).$$

We shall denote the weak convergence by $\Rightarrow$. Prohorov [1] has constructed a metric L in the space $M(R)$, such that the condition $\mu_n \Rightarrow \mu$ holds if and only if $L(\mu_n, \mu) \to 0$ $(\mu_n, \mu \in M(R))$.

The following theorems give — for various spaces R — the conditions for $\mu_n \Rightarrow \mu$ $(\mu_n, \mu \in M(R))$ expressed in the terms of convergence of certain measures induced by measures μ_n in finite dimensional Euclidean spaces.

Let $f_1(x)$, $f_2(x)$, ... be a fixed sequence of continuous mappings of R into a real line, such that for every $x \in R$: $\sup_n |f_n(x)| < \infty$, and let $\varphi_k(x) = \{f_1(x), ..., f_k(x)\}$. For $\mu \in M(R)$ let μ^k denote the measure induced in k-dimensional space R_k by mapping φ_k, i. e. let $\mu^k(A) = \mu\{\varphi_k^{-1}(A)\}$ for $A \subset R_k$ and $\varphi_k^{-1}(A) \in \mathcal{B}$. We shall assume that the metric in the space R_k is defined as

$$\varrho_k(\{u_1, ..., u_k\}, \{v_1, ..., v_k\}) = \max_{1\leqslant j\leqslant k} |u_j - v_j|.$$

Denote $\varrho^*(x, y) = \sup |f_n(x) - f_n(y)|$.

The following theorems hold:

THEOREM 1. *Let* $\mu_n \in M(R)$, $n = 0, 1, ...$. *If* $\varrho(x_m, x) \to 0$ *implies* $\varrho^*(x_m, x) \to 0$, *(i. e. the functions* $f_n(x)$ *are equicontinuous at every* $x \in R$*), then the condition*

$$\lim_{n\to\infty} \sup_k L(\mu_n^k, \mu_0^k) = 0 \tag{1}$$

is necessary for the condition $\mu_n \Rightarrow \mu_0$.

THEOREM 2. *Let* $\mu_n \in M(R)$, $n = 0, 1, \ldots$. *If* $\varrho^*(x_m, x) \to 0$ *implies* $\varrho(x_m, x) \to 0$ *then* (1) *is sufficient for* $\mu_n \Rightarrow \mu_0$.

Let now $D[0,1]$ denote the space of all real functions $\xi(t)$, $0 \leqslant t \leqslant 1$, whose discontinuities, if any, are of the first kind only and such that one of the equalities $\xi(t) = \xi(t-0)$ or $\xi(t) = \xi(t+0)$ holds. The space $D[0,1]$ with the metric d defined by Prohorov [1] is a separable complete metric space. Denote by $C[0,1]$ the space of all real continuous functions $\xi(t)$, $0 \leqslant t \leqslant 1$ with the uniform metric. For $\mu \in M(D|0,1|)$, or $\mu \in M(C|0,1|)$ denote by $\mu^{t_1, t_2, \ldots, t_m}$ the measure induced in R_k by the mapipng

$$\varphi_{t_1, t_2, \ldots, t_m}(\xi) = \{\xi(t_1 - 0),\ \xi(t_2 - 0, \ldots, \xi(t_m - 0)\}.$$

The following theorems hold:

THEOREM 3. *Let* $\mu_n \in M(D|0,1|)$, $n = 0, 1, \ldots$. *If the measure* μ_0 *has no fixed points of discontinuity (that is, for every fixed* t_0 *the set of those* $\xi \in D|0,1|$ *which are discontinuous at point* t_0 *is of* μ_0*-measure zero), then the condition*

$$\lim_{n \to \infty} \sup_{t_1, \ldots, t_m} L(\mu_n^{t_1, \ldots, t_m},\ \mu_0^{t_1, \ldots, t_m}) = 0$$

for every m, is necessary for the convergence $\mu_n \Rightarrow \mu_0$.

THEOREM 4. *Let* $\mu_n \in M(D|0,1|)$, $n = 0, 1, \ldots$. *If the measure* μ_0 *satisfies the condition* $\mu_0(C|0,1|^c) = 0$ (A^c *denotes the complement of* A), *then the condition*

$$\lim_{n \to \infty} \sup_{m} \sup_{t_1, \ldots, t_m} L(\mu_n^{t_1, \ldots, t_m},\ \mu_0^{t_1, \ldots, t_m}) = 0$$

is a necessary condition for the convergence $\mu_n \Rightarrow \mu_0$.

THEOREM 5. *Let* $\mu_n \in M(D|0,1|)$, $n = 0, 1, \ldots$. *The condition* (2) *is sufficient for the convergence* $\mu_n \Rightarrow \mu_0$.

As a consequence of Theorems 4 and 5 (as well as Theorems 1 and 2) we obtain

THEOREM 6. *Let* $\mu_n \in M(C|0,1|)$, $n = 0, 1, \ldots$. *The condition* (2) *is necessary and sufficient for the convergence* $\mu_n \Rightarrow \mu_0$.

The proofs of Theorems 1 — 6 will appear in paper [2].

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MATHEMATICS

Embedding of Two-Norm Spaces into the Space of Bounded Continuous Functions on a Half-Straight Line

by

Z. SEMADENI

Presented by W. ORLICZ on May 3, 1960

Introduction

This paper deals with universal two-norm spaces and with the negative solution of a problem of extension of γ-linear functionals from γ-closed subspaces.

Two spaces will be considered in particular: the space C_B of all bounded continuous real-valued functions $x(t)$ defined in $\langle 0, \infty)$, provided with the norms

$$|x| = \sup_{0 \leqslant t < \infty} |x(t)|, \qquad |||x|||^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{\langle 0, n \rangle} |x(t)|,$$

and the space CV of all real-valued continuous functions of bounded variation, defined in $\langle 0,1 \rangle$ and vanishing at $t = 0$, provided with

$$|x| = \operatorname*{Var}_{\langle 0,1 \rangle} x, \qquad |x|^* = \sup_{\langle 0,1 \rangle} |x(t)|.$$

Both spaces are normal, γ-complete and γ-separable *).

A two-norm space $\langle X, |\ |, |\ |^* \rangle$ will be said to be *γ-equivalent* to a space $\langle Y, \|\ \|, \|\ \|^* \rangle$, if there exists a one-to-one linear map of X onto Y, being an isometry with respect to $|\ |$ and an isomorphism with respect to $|\ |^*$. An analogous definition assuming isometry with respect to both norms does not seem to be appriopriate.

*) A two-norm space $\langle X, |\ |, |\ |^* \rangle$ is termed, successively: *normal*, if $|x_n - x|^* \to 0$ implies $|x| \leqslant \varliminf |x_n|$; *$\gamma$-complete*, if $x_p - x_q \xrightarrow{\gamma} 0$, *as* $p, q \to \infty$ implies existence of an $x \in X$ such that $x_n \xrightarrow{\gamma} x$ ($z_n \xrightarrow{\gamma} z$ means that $|z_n - z|^* \to 0$ and $\sup |z_n| < \infty$); *γ-separable*, if there exists a countable set $u_1, u_2, \ldots$, in X such that $u_{n_k} \xrightarrow{\gamma} x$ for every $x \in X$ and for a sequence n_k depending on x (or, equivalently, if $\langle X, |\ |^* \rangle$ is separable, see [2], p. 284). If $X_S = \{x \in X : |x| \leqslant 1\}$ is a Saks space, cf. [6], then γ-separability of X means separability of X_S. A functional ξ on $\langle X, |\ |, |\ |^* \rangle$ is called *γ-linear* if it is distributive and if $x_n \xrightarrow{\gamma} x_0$ implies $\xi(x_n) \to \xi(x_0)$.

Now, any linear subset X_0 of a two-norm space $\langle X, \|\,\|, \|\,\|^* \rangle$ is also a two-norm space with $\|\,\|$ and $\|\,\|^*$ induced by X.

The extension of γ-linear functionals on X_0 to γ-linear functionals on X is, in general, not possible There exist a normal two-norm space X, a linear subset X_0 and a γ-linear functional ξ_0 on X_0 which does not admit any γ-linear extension to the smallest γ-closed span of X_0 in X ([1], p. 133).

A. Alexiewicz has raised the following problem. Let X_0 be sequentially γ-closed in X and let X be normal. Does any γ-linear functional on X_0 admit a γ-linear extension on X? This question is answered in the negative by a suitable example constructed in this paper.

The main results of the paper are the following:

THEOREM 1. *The space C_B is universal for γ-separable normal two-norm spaces.*

More precisely: *for every normal γ-separable two-norm space $\langle X, \|\,\|, \|\,\|^* \rangle$ there exists a linear subset Y of C_B such that the space $\langle Y, |\,|, |\,|^* \rangle$ (with the norms induced by C_B) is γ-equivalent to $\langle X, \|\,\|, \|\,\|^* \rangle$.*

THEOREM 2. *There exists a subspace Y_0 of $\langle C_B, |\,|, |\,|^* \rangle$ γ-equivalent to $\langle CV, |\,|, |\,|^* \rangle$ and such that certain γ-linear functionals on $\langle Y_0, \|\,\|, \|\,\|^* \rangle$ do not admit any γ-linear extension to the whole space C_B.*

Two-norm spaces defined by a sequence of pseudonorms

Let Z be a linear space and let $\|\,\|_n$ be a sequence of (homogeneous) pseudonorms defined in Z and such that the conditions $x\|_n = 0$ $(n = 1, 2, \ldots)$ imply $x = 0$. The set **)

$$X = \{z \in Z : \sup |z|_n < \infty\}$$

is a normal two-norm space with the norms

$$|x| = \sup \|x\|_n \quad \text{and} \quad x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n .$$

It is well-known that many examples of two-norm spaces are defined in such a way and that the conditions

$$\sup_m \|x_m| < \infty \quad \text{and} \quad \lim_m \|x_m\|_n = 0 \quad \text{for} \quad n = 1, 2, \ldots,$$

imply $|\sum 2^{-n} (\|x_m\|_n)^p|^{1/p} \to 0$ for $p \geq 1$, therefore various starred norms give the same convergence γ. If Z is complete with respect to the metric

$$\varrho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n},$$

**) It may happen that X consists only of one element 0; this trivial case will be excluded.

then $\langle X, \|\,\|, \|\,\|^* \rangle$ is γ-complete; if $\langle Z, \varrho \rangle$ is separable, X is γ-separable. If $\langle Z, \varrho \rangle$ is complete and if the quotient spaces $\langle Z/\|\,\|_n, \|\,\|_n \rangle$ are prereflexive, then $\langle X, \|\,\|, \|\,\|^* \rangle$ is γ-reflexive ***), since the norm $\|\,\|^*$ gives the same convergence on the sphere $S = \{x : x \in X, \|x\| \leqslant 1\}$ as the norm

$$\|x\|^{**} = \left[\sum_{n=1}^{\infty} \frac{1}{2^n} (\|x\|_n)^2 \right]^{1/2}$$

and the space $\langle X, \|\,\|^{**} \rangle$ may be embedded into the l^2-product of prereflexive normed spaces $\langle X/\|\,\|_n, \|\,\|_n \rangle$ consisting of sequences $(x_1, x_2, \ldots)$ such that $\sum 2^{-n} (\|x_n\|_n)^2 < \infty$ (cf. [3], p. 192 and p. 243).

Let us notice that the topology of Z given by ϱ does not determine the space X uniquely; X depends essentially on the chosen pseudonorms $\|\,\|_n$. E.g. pseudonorms $\|x\|_n^{(1)} = |t_n|$ and $\|x\|_n^{(2)} = |t_1 + \cdots + t_{n+1}|$ determine the same topology in the space s of all real sequences $x = \{t_n\}$ and determine different two-norm spaces (m in the first case and l in the second).

PROPOSITION 1. *Given any normal two-norm space $\langle X, \|\,\|, \|\,\|^* \rangle$ satisfying condition $\|x\|^* \leqslant \|x\|$ for $x \in X$, there exists a sequence of pseudonorms $\|\,\|_n$ in X and a constant c such that*

$$\|x\| = \sup_n \|x\|_n \quad \text{for} \quad x \in X, \tag{1}$$

$$\|x\|^* \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n \leqslant c\,\|x\|^* \quad \text{for} \quad x \in X. \tag{2}$$

In other words, any normal two-norm space is γ-equivalent to a space constructed by the method described above.

Proof ****). Let Ξ and Ξ^* be the class of linear functionals on X, continuous with respect to $\|\,\|$ and $\|\,\|^*$, respectively, and let

$$\|\xi\| = \sup \{\xi(x) : \|x\| \leqslant 1,\ x \in X\}, \quad \|\xi\|^* = \sup \{\xi(x) : \|x\|^* \leqslant 1,\ x \in X\},$$

$$\Sigma = \{\xi : \xi \in \Xi,\ \|\xi\| \leqslant 1\}, \quad \Sigma^* = \{\xi \in \Xi^* : \|\xi\|^* \leqslant 1\},$$

$$\|x\|_n = \sup \{\xi(x) : \xi \in (n\Sigma^*) \cap \Sigma\}. \tag{3}$$

Obviously, $\|x\|_n \leqslant \|x\|_{n+1}$ for $n = 1, 2, \ldots$, and $\Sigma^* \subset \Sigma$, whence

$$\|x\|^* = \{\xi(x) : \xi \in \Sigma^*\} = \|x\|_1 \leqslant \|x\|_n \leqslant n\,\|x\|^*,$$

$$\|x\|^* \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n \leqslant \sum_{n=1}^{\infty} \frac{n}{2^n} \|x\|^* = c\,\|x\|^*,$$

***) A normed space is called *prereflexive*, if its completion is reflexive; a normal two-norm space is called *γ-reflexive* if every linear functional δ on $\langle \Xi^*, \|\,\| \rangle$ is of the form $\delta(\xi) = \xi(z)$ with a fixed $z \in X$ (see [2], p. 280). If $\langle X, \|\,\|^* \rangle$ is prereflexive, then $\langle X, \|\,\|, \|\,\|^* \rangle$ γ-reflexive, whenever it is γ-complete.

****) Recently, Professor A. Alexiewicz communicated to me that he had an independent proof of this proposition.

$$\sup |x|_n = \sup \{\xi(x) : \xi \in \bigcup_{n-1}^{\infty} (n\Sigma^*) \cap \Sigma\} = \sup \{\xi(x) : \xi \in \Xi^* \cap \Sigma\} = \|x\|$$

(by a theorem proved in [1], p. 124).

In the considered case, $\|\ \|_n$ are norms mutually equivalent and equivalent to $\|\ \|^*$. Thus, pseudonorms $\|\ \|_n$ satisfying (1) and (2) are not uniquely determined by $\langle X, |\ |, \|\ \|^* \rangle$ and the corresponding B_0^*-spaces may possess extremally different properties.

Now, by a theorem of S. Mazur and W. Orlicz ([4], p. 141) and by $\|x\|_n \leqslant |x|_{n+1}$, there exists a one-to-one linear operation U from X into C_B such that

$$\|x\|_n = \sup \{|(Ux)(t)| : 0 \leqslant t \leqslant n\}.$$

Thus, Theorem 1 is an immediate consequence of Proposition 1.

Embedding of the space CV into C_B

Theorem 1 establishes the existence of an embedding of any normal γ-separable two-norm space into the space C_B. However, this embedding, although effective, is not convenient in certain cases and does not coincide with the natural embedding in the typical cases when the pseudonorms generating a given two-norm space are not equivalent mutually. In this section we shall define a special embedding of CV into C_B, which will be needed in the proof of Theorem 2.

Given any positive integer m, let $f_k^m(t)$ $(k = 1, 2, \ldots, 2^m)$ be a sequence of all right-hand continuous functions defined in $\langle 0,1\rangle$, constant on each interval $\left(\frac{l-1}{m}, \frac{l}{m}\right)$ $(l = 1, 2, \ldots, m)$, assuming only two values $+1$ and -1 for $0 \leqslant t < 1$, and vanishing at $t = 1$. Now, let $x \in CV$ and let

$$x_k^m(t) = \int_0^t f_k^m(\tau)\, dx(\tau).$$

Obviously,

$$|x_k^m(t)| \leqslant \sup_{\langle 0,t\rangle} |f_k^m(\tau)| \cdot \operatorname{Var}_{\langle 0,t\rangle} x \leqslant \operatorname{Var}_{\langle 0,1\rangle} x,$$

and

$$x_k^m(1) = \int_0^1 f_k^m(t)\, dx(t) = -\int_0^1 x(t)\, d f_k^m(t) =$$

$$= \sum_{l=1}^{m} x\left(\frac{l}{m}\right)\left| f_k^m\left(\frac{l}{m}\right) - f_k^m\left(\frac{l-1}{m}\right)\right| - \sum_{l=0}^{m-1} f_k^m\left(\frac{l}{m}\right)\left|x\left(\frac{l}{m}\right) - x\left(\frac{l+1}{m}\right)\right|,$$

whence $\sup_{k=1,\ldots,2^m} |x_k^m(1)| \leqslant \sum_{l=0}^{m-1} \left|x\left(\frac{l+1}{m}\right) - x\left(\frac{l}{m}\right)\right|$ and, by continuity of $x(t)$,

$$\sup_{m=1,2,\ldots} \sup_{k=1,\ldots,2^m} \sup_{0 \leqslant t \leqslant 1} |x_k^m(t)| = \operatorname{Var}_{\langle 0,1\rangle} x. \tag{4}$$

Now, let us set $f_n(t) = f_k^m(t)$ and $x_n(t) = x_k^m(t)$, where $n = 2^m + k - 2$, and

$$x_+(t) = \begin{cases} x_n(t) & \text{for } t \in \langle 2n-2,\ 2n-1\rangle,\ n = 1, 2, \ldots, \\ \text{linear on } \langle 2n-1,\ 2n\rangle, & n = 1, 2, \ldots. \end{cases}$$

PROPOSITION 2. *The map $x \to x_+$ is a γ-equivalence between $\langle CV, \|\ \|, \|\ \|^*\rangle$ and a subspace Y_0 of $\langle C_B, \|\ \|, \|\ \|^*\rangle$.*

Proof. Isometry with respect to $\|\ \|$ is given by (4), isomorphism with respect to $\|\ \|^*$ is evident.

Functionals on CV induced by γ-linear functionals on C_B

Every γ-linear functional on $\langle CV, \|\ \|, \|\ \|^*\rangle$ can be uniquely written in the form

$$\zeta_h(x) = \int_0^1 x(t)\, dh(t) = -\int_0^1 h(t)\, dx(t), \tag{5}$$

where $h(t)$ is a real-valued function defined in $\langle 0, 1\rangle$ and satisfying the following conditions ([1], p. 138):

$$h(t) \text{ is bounded and right-continuous, and } h(1) = 0; \tag{6}$$

$$\begin{cases} h(t) \text{ possesses only discontinuities of the first kind at points } t_1, t_2, \ldots, \\ \text{say, and } h(t_n) - h(t_n - 0) \to 0 \text{ as } n \to \infty. \end{cases} \tag{7}$$

Given a function $h(t)$ satisfying (6) and (7), let $\xi_h(x_n) = \int_0^1 x(t)\, dh(t)$ denote the corresponding γ-linear functional on $\langle Y_0, \|\ \|, \|\ \|^*\rangle$, where x and Y_0 were defined above.

PROPOSITION 3. *If a γ-linear functional ξ_h on $\langle Y_0, \|\ \|, \|\ \|^*\rangle$ corresponding to a functional* (5), *admits an extension on the whole space $\langle C_B, \|\ \|, \|\ \|^*\rangle$ to a γ-linear functional ξ, then*

$$\sum_{t \in \mathcal{K}} |h(t) - h(t-0)| < \infty, \tag{8}$$

where $\mathcal{K}$ denotes the set of irrational numbers
(the summation is extended on a countable set of values different from 0).

Proof. Let Y_1 be the subspace of C_B consisting of all functions linear on each interval $\langle 2n+1,\ 2n+2\rangle$ and vanishing at $t_n = 2n$ $(n = 0, 1, 2, \ldots)$. Evidently, $\langle Y_1, \|\ \|, \|\ \|^*\rangle$ is γ-equivalent to the space of all bounded continuous functions defined on $\bigcup_{n=0}^{\infty} \langle 2n, 2n+1\rangle$ and vanishing at $2n$, and it is γ-equivalent to the space of all sequences of uniformly

bounded continuous functions defined in $\langle 0, 1\rangle$ and vanishing at $t = 0$. Thus the functional ξ, considered only on Y_1, has the form

$$\xi(z) = \sum_{n=1}^{\infty} \int_0^1 z_n(t)\, dg_n(t),$$

where g_n are right-continuous, $g_n(1) = 0$ and $\sum_{n=1}^{\infty} \operatorname{Var}_{\langle 0,1\rangle} g_n < \infty$ cf. [5] and [1], p. 135). In particular, this formula is valid if $x(t)$ is any absolutely continuous function, $x_n(t)$ and $x_\#(t)$ being defined as in preceding section. Then

$$x_n(t) = \int_0^t f_n(\tau)\, x'(\tau)\, d\tau, \qquad \xi_h(x_\#) = -\int_0^1 h(t)\, x'(t)\, dt,$$

$$\xi(x_\#) = \sum_{n=1}^{\infty} \int_0^1 x_n(t)\, dg_n(t) = -\sum_{n=1}^{\infty} \int_0^1 g_n(t)\, x'(t) f_n(t)\, dt.$$

Since $\sum_{n=1}^{\infty} \sup_{0\leqslant t\leqslant 1} |f_n(t)\, g_n(t)| \leqslant \sum_{n=1}^{\infty} \operatorname{Var} g_n < \infty$, the series $\sum_{n=1}^{\infty} f_n(t)\, g_n(t) = g(t)$ is uniformly convergent, whence

$$\xi(x_\#) = -\int_0^1 x'(t) \sum_{n=1}^{\infty} g_n(t)\, f_n(t)\, dt = -\int_0^1 g(t)\, x'(t)\, dt.$$

Assuming $\xi(x_\#) = \xi_h(x_\#)$ for $x_\# \in Y_0$ and comparing the foregoing formulas, we infer $g(t) = h(t)$ almost everywhere. Since both functions are right-continuous and vanishing at $t = 1$, $g(t) = h(t)$ for all $t \in \langle 0, 1\rangle$. Finally,

$$\sum_{t\in\mathcal{K}} |g(t) - g(t-0)| \leqslant \sum_{t\in\mathcal{K}} \sum_{n=1}^{\infty} |g_n(t) - g_n(t-0)| \leqslant \sum_{n=1}^{\infty} \operatorname{Var} g_n < \infty.$$

Thus, we have proved Proposition 3. To deduce Theorem 2 from Propositions 2 and 3 let us observe that conditions (6) and (7) do not imply (8).

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MATHEMATICS

Extension of Linear Functionals in Two-Norm Spaces

by

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The main result of this paper consists in establishing that if $\langle X_0, \|\ \|, \|\ \|^* \rangle$ is a γ-reflexive subspace of a normal two-norm space $\langle X, \|\ \|, \|\ \|^* \rangle$, then every γ-linear functional on X_0 has a γ-linear extension on the whole space X.

Preliminaries

In the present paper X will denote a linear space provided with two (homogeneous) norms $\|\ \|$ and $|\ |^*$, with $|x|^* \leqslant \|x\|$. A sequence x_n of elements of X is called γ-convergent to x_0 (written $x_n \xrightarrow{\gamma} x_0$) if $\lim \|x_n - x_0\|^* = 0$ and $\sup \|x_n\| < \infty$. Ξ and Ξ^* will denote the set of all linear functionals (on X) continuous with respect to $\|\ \|$ and $|\ |^*$, respectively, and Ξ^γ — the set of γ-linear functionals (sequentially continuous). Next, we shall denote (for $\xi \in \Xi$ or $\xi \in \Xi^*$):

$$\|\xi\| = \sup\{|\xi(x)| : x \in X, \ \|x\| \leqslant 1\}, \qquad |\xi|^* = \sup\{|\xi(x)| : x \in X, \ |x|^* \leqslant 1\},$$

$$\Sigma = \{\xi : \xi \in \Xi, \ \|\xi\| \leqslant 1\}, \qquad S = \{x : x \in X, \ |x| \leqslant 1\}.$$

We have $\Xi^* \subset \Xi^\gamma \subset \Xi$ and $|\xi| \leqslant \|\xi\|^*$. A subset A of X is called γ-closed, if $x_n \in A$, $x_n \xrightarrow{\gamma} x$ imply $x \in A$.

A two-norm space $\langle X, \|\ \|, |\ |^* \rangle$ is called *quasi-normal with a constant* K, if $\|x_n - x\|^* \to 0$ implies $\|x\| \leqslant K \varliminf \|x_n\|$. This condition will be assumed in the sequel; it is equivalent to the existence of a norm $\|\ \|_1$ such that $\|x\|_1 \leqslant \|x\| \leqslant K\|x\|_1$ and such that $\langle X, \|\ \|_1, \|\ \|^* \rangle$ is normal (i. e. quasi-normal with $K = 1$, [4], p. 118). $\langle \Xi^*, \|\ \|^*, \|\ \| \rangle$ is a γ-complete *)

*) $\langle X, \|\ \|, \|\ \|^* \rangle$ is called *γ-complete*, if $x_p - x_q \xrightarrow{\gamma} 0$ as $p, q \to \infty$ implies existence of an element $x \in X$ such that $x_n \xrightarrow{\gamma} x$, and is called *$\gamma$-compact* if every sequence x_n bounded with respect to $\|\ \|$ has a γ-convergent subsequence. If X is normal, γ-completeness means that $X_S = \{x : x \in X, \ \|x\| \leqslant 1\}$ is a *Saks space*, *cf.* [8], and γ-compactness means that X_S is compact.

normal two-norm space, called the *γ-conjugate* of $\langle X, \|\ \|, \|\ \|^* \rangle$. The space $\langle \mathscr{X}^{(\gamma)}, |\ |, \|\ \|^* \rangle$, γ-conjugate to $\langle \Xi^*, \|\ \|^*, \|\ \| \rangle$, is defined analogously. The canonical embedding $\hat{x}(\xi) = \xi(x)$ *for* $x \in X$ maps X into $\mathscr{X}^{(\gamma)}$ isomorphically with respect to both norms $\|\ \|$ and $\|\ \|^*$ ([3], p. 279 and [4], p. 119). $\langle X, \|\ \|, \|\ \|^* \rangle$ is called γ-reflexive, if the canonical embedding maps X onto $\mathscr{X}^{(\gamma)}$ or, equivalently, if every linear functional $\hat{}$ on $\langle \Xi^\gamma, \|\ \| \rangle$ is of the form $\hat{}(\xi) = \xi(z)$ with a fixed $z \in X^*$).

Now, let X_0 be a linear subset of X, let $S_0 = \{x : x \in X_0,\ \|x\| \leqslant 1\}$ and let Ξ_0, Ξ_0^*, Ξ_0^γ, Σ_0, $\mathscr{X}_0^{(\gamma)}$, etc. denote the classes of functionals defined for X_0 (instead of X) in a way analogous to the above.

We shall say that X_0 has the property (W_∞), if every γ-linear functional ξ_0 defined on X has a γ-linear extension to the whole space X. There exists a normal, γ-compact space $\langle X, |\ |, |\ |^* \rangle$, a linear subset X_0 of X and a γ-linear functional ξ_0 on X_0 which cannot be extended to a γ-linear functional defined on the smallest γ-closed span of X_0 ([1], p. 133). However, in any case every γ-linear functional may be extended to a γ-linear functional on the closure with respect to $\|\ \|$ ([1], p. 124). Next, there exist a normal γ-complete space $\langle X, |\ |, |\ |^* \rangle$, a γ-closed subspace X_0 of X and a γ-linear functional on X_0 which do not admit any γ-linear extension on X, see [10].

Some sufficient conditions are known for the property (W_∞): if X_0 is a lattice-ideal of a two-norm lattice X ([2], p. 138), or if $\langle X, |\ |, |\ |^* \rangle$ is normal and γ-compact and X_0 is closed with respect to $\|\ \|^*$ (Wiweger [11], Theorem 2.6.5), or if $\langle X, |\ | \rangle$ is reflexive and X_0 is any linear subset (in this case $\Xi^\gamma = \Xi$ and $\Xi_0^\gamma = \Xi_0$, [3], p. 281), then X_0 has the property (W_∞). Finally, the following two conditions are obvious. If there exists a γ-γ-linear projection P of X onto X_0 (i.e. a distributive operation P such that $P^2 = P$, $P(X) = X_0$, and such that $x_n \xrightarrow{\gamma} 0$ implies $P(x_n) \xrightarrow{\gamma} 0$), then the adjoint operation $\eta = P^*(\xi)$, where $(P^*\xi)(x) = \xi(Px)$, establishes a γ-linear simultaneous extension of all functionals of Ξ_0^γ. If $\|\ \|$ and $\|\ \|^*$ are equivalent on X_0, then X_0 has the property (W_∞), too.

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*) This definition is more general than in [3]; (cf. [4], p. 119).

Each of the following conditions is sufficient for a normal space $\langle X, \|\ \|, \|\ \|^* \rangle$ to be γ-reflexive, [3]:

(i) $\langle X, \|\ \|, \|\ \|^* \rangle$ *is* γ*-compact,*

(ii) $\langle X, \|\ \| \rangle$ *is reflexive,*

(iii) $\langle X, \|\ \|, \|\ \|^* \rangle$ *is* γ*-complete and there exists a norm* $\|\ \|_1^*$ *leading to the same convergence* γ *as* $\|\ \|^*$ *and such that the completion of* $\langle X, \|\ \|_1^* \rangle$ *is reflexive.*

Thus, the assumption of γ-reflexivity is not so strong as the analogous assumption of reflexivity of a Banach space. Most of known examples of two-norm spaces satisfy at least one of these conditions.

The property (W_K)

Let $\langle X, \|\ ,|\ |^* \rangle$ be a two-norm space and let X_0 be a linear subset of X. We shall say that X_0 *has the property* (W_K), K being a constant, if for every γ-linear functional $\xi_0(x)$ defined on X_0 there exists a γ-linear functional $\xi(x)$ defined on X and such that

$$\xi(x) = \xi_0(x) \quad \text{for } x \in X_0, \tag{1}$$

$$\|\xi\| \leqslant K \|\xi_0\|. \tag{2}$$

THEOREM 1. *Let X_0 be any linear subset of any two-norm space $\langle X, \|\ ,|\ |^* \rangle$. If X_0 has the property (W_∞), then it has the property (W_K) for a certain K.*

Proof. Let us consider the operation of restriction $\xi_0 = U(\xi)$, where ξ_0 is determined by (1). U is a linear operation from $\langle \Xi^\gamma, \|\ \| \rangle$ into $\langle \Xi_0^\gamma, \|\ \| \rangle$. Condition (W_∞) means that U maps Ξ^γ onto Ξ_0^γ; $\langle \Xi^\gamma, \|\ \| \rangle$ and $\langle \Xi_0^\gamma, \|\ \| \rangle$ being complete ([9], p. 57), there exists $r > 0$ such that

$$U(\Xi^\gamma \cap \Sigma) \supset \Xi_0^\gamma \cap r\Sigma_0$$

(by a theorem of Banach, [5], p. 38). In turn, the latter condition means that X_0 has the property $(W_{1/r})$.

THEOREM 2. *In order that X_0 have the property $(W_{K+\varepsilon})$ for all $\varepsilon > 0$, it is necessary and sufficient that the inequality*

$$\sup\{\vartheta(\xi_0) : \xi \in \Xi^\gamma \cap \Sigma\} \geqslant \frac{1}{K} \sup\{\vartheta(\eta) : \eta \in \Xi_0^\gamma \cap \Sigma_0\} \tag{3}$$

be satisfied for every $\vartheta \in \mathfrak{A}_0^{(\gamma)}$, ξ_0 being the restriction of ξ defined by (1).

Proof. This follows by Banach's theorem on adjoint operations ([5], p. 146) applied to the operation U considered in the former proof. One may verify that the constant K is the same in both cases (e.g. analysing the proof of Banach's theorem given by S. Mazur, [6], p. 34).

Now, K being a positive constant, let us consider another condition:

(V_K) *for every $\zeta \in \Xi_0^* \cap \Sigma_0$ there exists a constant L such that the inequality*

$$\zeta(x) < L\,|y|^* + K\,|x + y|$$

holds for every $x \in X_0$ and $y \in X$.

THEOREM 3. *Condition (V_K) implies $(W_{K+\varepsilon})$, ε being any positive number.*

Proof. Let $\zeta \in \Xi_0^* \cap \Sigma_0$. Then, for every $\lambda_1 \geqslant 0, \ldots, \lambda_n \geqslant 0$, $\mu_1 \geqslant 0, \ldots, \mu_m \geqslant 0$, $x_1 \in X_0, \ldots, x_n \in X_0$, $y_1 \in X, \ldots, y_m \in X$, the inequality (V_K) implies the following:

$$\sum_{i=1}^{n} \lambda_i \zeta(x_i) - L \sum_{j=1}^{m} \mu_j \|y_j\|^* \leqslant K \left\| \sum_{i=1}^{n} \lambda_i x_i + \sum_{j=1}^{m} \mu_j y_j \right\|.$$

Hence, by a theorem of S. Mazur and W. Orlicz ([7], p. 147), there exists a distributive functional ξ satisfying the folowing inequalities

$$\xi(x) \geqslant -L\|x\|^* \quad \text{for} \quad x \in X,$$

$$\xi(x) \geqslant \zeta(x) \quad \text{for} \quad x \in X_0,$$

$$\xi(x) \leqslant K\|x\| \quad \text{for} \quad x \in X.$$

In other words, for every $\zeta \in \Xi_0^* \cap \Sigma_0$ there exists a functional $\xi \in \Xi^*$ such that $\xi(x) = \zeta(x)$ for $x \in X_0$ and $\|\xi\| \leqslant K$. Let us notice that the converse implication is also true: the last statement implies (V_K).

Now, let $\xi_0 \in \Xi_0^\gamma$ and let $\xi_0 = \sum_{n=1}^{\infty} \zeta_n$ with $\zeta_n \in \Xi_0^*$ and $\sum_{n=1}^{\infty} \|\zeta_n\| \leqslant (1+\varepsilon K^{-1})\|\xi_0\|$ ([1], p. 132). By the foregoing arguments, there exist functionals $\xi_n \in \Xi^*$ such that $\xi_n(x) = \zeta_n(x)$ for $x \in X_0$ and $\|\xi_n\| \leqslant K\|\zeta_n\|$, $n = 1, 2, \ldots$ Obviously, the functional $\xi = \sum \xi_n$ is a γ-linear extension of ξ_0 and

$$\|\xi\| \leqslant \sum_{n=1}^{\infty} \|\xi_n\| \leqslant K \sum_{n=1}^{\infty} \|\zeta_n\| \leqslant (K+\varepsilon)\|\xi_0\|.$$

THEOREM 4. *Let $\langle X, |\,,\,|, \|\,\|^* \rangle$ be normal, let $\langle X_0, |\,,\,| \rangle$ be complete and let $\varepsilon > 0$. Now, let us assume that, for every $\zeta \in \Xi_0^* \cap \Sigma$, there exist $L_0 > 0$ and $\delta > 0$ such that the inequality*

$$\zeta(x) \leqslant L_0\|y\|^* + (1+\varepsilon)|x+y|$$

is satisfied whenever $\|y\|^* < \delta$ and $x \in S_0$. Then X_0 has the property $(V_{1,\varepsilon})$.

Proof. Assumption $x \in S_0$ is not essential. Let $\varepsilon > 0, \zeta \in \Xi_0^* \cap \Sigma_0$ and let

$$A_L = \bigcap_{y \in X} \{x : x \in X_0,\ \zeta(x) \leqslant L|y|^* + (1+\varepsilon)|x+y|\},$$

$$\|x\|_L^K = \inf\{L\|y\|^* + K|x+y| : y \in X\} \quad \text{for} \quad x \in X_0,$$

K and L being positive constants. One may verify that $|x|_L^K$ is a norm equivalent to $|\,|$, $\min(K, L) \cdot |x| \leqslant |x|_L^K \leqslant L\|x\|^*$ and $|x|_L^K \leqslant K|x|$. We shall prove that

$$\lim_{L \to \infty} |x|_L^K = K|x| \quad \text{for any} \quad x \in X_0.$$

Indeed, $\lim_{L \to \infty} |x|_L^K \leqslant K|x|$; conversely, for every L there exists $y_L \in X$ such that $|x|_L^K > L\|y_L\|^* + K|x+y_L| - L^{-1}$. If $L \to \infty$, then

$$0 \leqslant \|y_L\|^* < L^{-1}|x|_L^K - KL^{-1}|x+y_L| + L^{-2} \leqslant L^{-1}K|x| + L^{-2}$$

implies $\|y_L\|^* \to 0$. Since $\langle X, |\,|, \|\,\|^* \rangle$ is normal,

$$\lim_{L \to \infty} |x|_L^K \geqslant \lim_{L \to \infty} |L\|y_L\|^* + K|x+y_L|| \geqslant K \lim_{L \to \infty} |x+y_L| = K|x|.$$

Hence, $\lim_{L\to\infty} \|x\|_L^{1+\varepsilon} = (1+\varepsilon)\ \|x\| > |x| \cdot \zeta(x)$ for $x \neq 0$, and $\bigcup_{L=1} A_L = X_0$. Obviously, A_L are closed in X_0, $\|\ \|$. Let us notice that A_L and $A_L \cap S_0$ are also closed with respect to $\|\ \|^*$, for

$$A_L = \bigcap_{y \in X} \{x : x \in X_0,\ \zeta(x) \leqslant L\,|x+y|^* + (1+\varepsilon)\,|y|\}.$$

Therefore, by the theorem of Baire, there exist $x_0 \in X_0$, $r > 0$, and $L_1 > 0$ such that the conditions $\|u - x_0\| \leqslant r$, $u \in X_0$, $z \in X$ imply

$$\zeta(u) \leqslant L_1 \|z\|^* + (1+\varepsilon)\,\|u+z\|. \tag{4}$$

Next, there exists $L_2 > 0$ such that

$$\zeta(x_0) \geqslant -L_2 \|y\|^* + (1+\varepsilon)\,\|x_0\|, \tag{5}$$

whenever $\|y\|^* \geqslant \delta$. Let $z = ry$, $x \in S_0$, $L = \max(L_0, L_1 + L_2 r^{-1})$, $u = x_0 + rx$. Then, by (4) and (5),

$$\zeta(x) = r^{-1}\zeta(u) - r^{-1}\zeta(x_0) \leqslant r^{-1} L_1 \|yr\|^* + r^{-1}(1+\varepsilon)\|x_0 + rx + ry\| +$$
$$+ L_2 r^{-1} \|y\|^* - r^{-1}(1+\varepsilon)\|x_0\| \leqslant L\,|y|^* + (1+\varepsilon)\,|x+y|,$$

whenever $\|y\|^* \geqslant \delta$. The above inequality is assumed for $\|y\|^* < \delta$. Thus X_0 satisfies condition $(V_{1+\varepsilon})$.

Case of γ-reflexive spaces

In this section a useful sufficient condition will be established.

THEOREM 5. *Let $\langle X, \|\ \|, |\ |^* \rangle$ be a two-norm space quasi-normal with a constant K, and let X_0 be any of its linear subsets such that $\langle X_0, \|\ \|, \|\ \|^* \rangle$ is γ-reflexive with respect to norms $\|\ \|$ and $\|\ \|^*$ induced by X. Then, for each $\varepsilon > 0$, X_0 has the property $(W_{K+\varepsilon})$.*

This theorem is a generalization of the quoted theorem of Wiweger.

Proof. Applying Proposition 1.1 of [4] we reduce our considerations to the case $K = 1$, i.e. we may assume $\langle X, \|\ \|, |\ |^* \rangle$ to be normal. By Theorem 2, we have to prove the inequality

$$\sup\{\xi(\xi_0) : \xi \in \Xi^\gamma \cap \Sigma\} \geqslant \|\xi\| \quad \text{for all} \quad \xi \in X_0^{(\gamma)}. \tag{6}$$

Now, X_0 being γ-reflexive, every functional $\xi \in X_0^{(\gamma)}$ has the form

$$\xi(\xi_0) = \xi_0(z) \quad \text{for} \quad \xi_0 \in \Xi_0^\gamma$$

with a fixed $z \in X_0$. Since $\xi_0(z) = \xi(z)$, (6) may be written equivalently:

$$\sup\{\xi_0(z) : \xi \in \Xi^\gamma \cap \Sigma\} = \sup\{\xi(z) : \xi \in \Xi^\gamma \cap \Sigma\} \geqslant \|\xi\| = \|z\|.$$

This condition was proved in [1], p. 124.

THEOREM 6. Let $\langle X, \|\|, \|\|^* \rangle$ *be normal and* γ*-reflexive, and let* X_0 *be any* γ*-closed linear subset of* X*. Then, for each* $\varepsilon > 0$, X_0 *has the property* $(W_{1+\varepsilon})$.

Proof. Any γ-closed subspace of a γ-reflexive space is γ-reflexive, too ([3], p. 281); thus, Theorem 6 is an immediate consequence of Theorem 5 *).

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*) Recently Dr A. Wiweger proved Theorem 6 (without estimation of norms) in a quite different way analogous to that used in [11].

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MATHEMATICS

A Generalization of Chebyshev Polynomials

by

Jan MYCIELSKI and S. PASZKOWSKI

Presented by E. MARCZEWSKI on May 5, 1960

1. It is the purpose of this paper to prove the following theorem.

THEOREM. *Given $n+1$ positive real numbers $w_0, w_1, ..., w_n$ there exists one and only one polynomial $p(t)$ of degree n with real coefficients for which there are numbers $-1 = v_0 < v_1 < \cdots < v_n = 1$ such that*

$$p(v_k) = (-1)^{n-k} w_k \quad (k = 0,1, ..., n)$$

and

$$\frac{d}{dt} p(v_l) = 0 \quad (l = 1, ..., n-1).$$

This is well known in the case $w_0 = w_1 = \cdots = w_n = 1$. The polynomials of Chebyshev

$$(*) \qquad p(t) = \cos(n \arccos t)$$

are the solution. The Theorem in its general form is needed in the approximation theory, [2], [3]. We do not know any explicit expression of our polynomials comparable to (*). The proof which follows in sections **2** — **4** of this paper uses a topological method and does not give any such formula. Some methods for computation of the values of our polynomials and some tables of values in some special cases are given with applications in [3].

2. Let P be the set of points $(c, z_1, ..., z_n) \in R^{n+1}$ satisfying

$$c > 0, \quad -1 < z_1 < \cdots < z_n < 1;$$

and W the set of points $(w_0, w_1, ..., w_n) \in R^{n+1}$ satisfying

$$w_i > 0 \quad (i = 0,1, ..., n).$$

We will define a mapping $\varphi : P \to W$. Let

$$(1) \qquad p(t) = c \prod_{i=1}^{n} (t - z_i).$$

Then the derivative dp/dt has real zeros $v_1, ..., v_{n-1}$ and

$$z_1 < v_1 < z_2 < \cdots < v_{n-1} < z_n.$$

We put $\varphi(c, z_1, ..., z_n) = (w_0, w_1, ..., w_n)$, where

$$w_j = (-1)^{n-j}\, p(v_j) \qquad (v_0 = -1,\ v_n = 1,\ j = 0,1, ..., n). \tag{2}$$

It is clear that the Theorem is equivalent to the following statement

(T) *The mapping* $\varphi : P \to W$ *is* $1-1$ *and onto.*

Proof. It is obvious that P and W are homeomorph with R^{n-1}. Let us consider the one point compactifications $P \cup (\infty_1)$ and $W \cup (\infty_2)$. Therefore, $P \cup (\infty_1)$ and $W \cup (\infty_2)$ are homeomorph with the $(n+1)$-dimensional sphere S^{n+1}. We put $\varphi(\infty_1) = \infty_2$.

The following two lemmas will be proved in sec. **3** and **4**.

(L_1) *The mapping* $\varphi : P \to W$ *is a local homeomorphism.*

(L_2) *The mapping* $\varphi : P \cup (\infty_1) \to W \cup (\infty_2)$ *is continuous.*

By a theorem proved in [1] the above facts imply that φ is a homeomorphism of $P \cup (\infty_1)$ onto $W \cup (\infty_2)$. Since $\varphi(\infty_1) = \infty_2$, we obtain (T).

3. It is clear that (L_1) is a consequence of the following:

LEMMA 1. *The Jacobi determinant of* φ *on* P *has no zeros.*

Proof. By (1) and (2) we have

$$w_j = (-1)^{n-j}\, c \prod_{i=1}^{n} (v_j - z_i).$$

The points v_0 and v_n are constant, the remaining points v_j are dependent on z_i but not on c. Therefore,

$$\frac{\partial w_j}{\partial c} = (-1)^{n-j} \prod_{i=1}^{n} (v_j - z_i) = \frac{w_j}{c},$$

$$\frac{1}{w_j}\frac{\partial w_j}{\partial z_r} = \sum_{i=1}^{n} \frac{\dfrac{\partial v_j}{\partial z_r} - \delta_{ir}}{v_j - z_i} = \frac{\partial v_j}{\partial z_r} \sum_{i=1}^{n} \frac{1}{v_j - z_i} - \frac{1}{v_j - z_r} = -\frac{1}{z_r - v_j}$$

$$(r = 1, ..., n).$$

The last equality follows from that

$$\frac{\partial v_j}{\partial z_r} = 0 \text{ for } j = 0 \text{ and } j = n, \quad \sum_{i=1}^{n} \frac{1}{v_j - z_i} = \frac{p'(v_j)}{p(v_j)} = 0 \text{ for } j = 1, ..., n-1.$$

Therefore,

$$\frac{\partial(w_0, w_1, \ldots, w_n)}{\partial(c, z_1, \ldots, z_n)} = \begin{vmatrix} \frac{w_0}{c} & \frac{w_0}{z_1 - v_0} & \frac{w_0}{z_2 - v_0} & \cdots & \frac{w_0}{z_n - v_0} \\ \frac{w_1}{c} & \frac{w_1}{z_1 - v_1} & \frac{w_1}{z_2 - v_1} & \cdots & \frac{w_1}{z_n - v_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{w_n}{c} & \frac{w_n}{z_1 - v_n} & \frac{w_n}{z_2 - v_n} & \cdots & \frac{w_n}{z_n - v_n} \end{vmatrix} =$$

$$= \frac{w_0 w_1 \ldots w_n}{c} \begin{vmatrix} 1 & \frac{1}{z_1 - v_0} & \frac{1}{z_2 - v_0} & \cdots & \frac{1}{z_n - v_0} \\ 1 & \frac{1}{z_1 - v_1} & \frac{1}{z_2 - v_1} & \cdots & \frac{1}{z_n - v_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \frac{1}{z_1 - v_n} & \frac{1}{z_2 - v_n} & \cdots & \frac{1}{z_n - v_n} \end{vmatrix}.$$

Developping this determinant with respect to the first column and using known formulas for Cauchy's determinants we obtain

$$\frac{\partial(w_0, w_1, \ldots, w_n)}{\partial(c, z_1, \ldots, z_n)} = (-1)^{n(n-1)/2} \frac{w_0 w_1 \ldots w_n}{c} \sum_{j=0}^{n} \frac{(-1)^j z^{\Delta} v_j^{\Delta}}{\prod_{i=1}^{n} \prod_{\substack{l=0 \\ l \neq j}}^{n} (z_i - v_l)}, \tag{3}$$

where z^{Δ} is the determinant of Vandermonde of the numbers $z_1, \ldots, z_n$, and v_j^{Δ} is the determinant of Vandermonde of the numbers $v_0, \ldots, v_{j-1}$, $v_{j+1}, \ldots, v_n$. Both these determinants are positive. Since

$$\prod_{i=1}^{n} \prod_{\substack{l=0 \\ l \neq j}}^{n} (z_i - v_l) = \frac{\prod_{i=1}^{n} \prod_{l=0}^{n} (z_i - v_l)}{\prod_{i=1}^{n} (z_i - v_j)} = \frac{\prod_{i=1}^{n} \prod_{l=0}^{n} (z_i - v_l)}{(-1)^j \prod_{i=1}^{n} |z_i - v_j},$$

all the terms of the sum in (3) are of the same sign. and the sum is different from zero.

4. It is clear that (L_2) is a consequence of the following:

LEMMA 2. *If* $0 < d \leqslant 1$ *and*

$$\min\left(c, \frac{1}{c}, z_1 + 1, z_2 - z_1, \ldots, z_n - z_{n-1}, 1 - z_n\right) \leqslant d,$$

then

$$\min\left(w_0, \frac{1}{w_0}, w_1, \frac{1}{w_1}, \ldots, w_n, \frac{1}{w_n}\right) \leqslant K_n \sqrt{d},$$

where K_n *depends only on* n.

For proving this lemma we firstly need the following:

LEMMA 3. *If the polynomial $p(t)$ of degree n has n real zeros $z_1 < z_2 < \dots < z_n$ and dp/dt has the zeros $v_1 < v_2 < \dots < v_{n-1}$, then*

$$z_j + \frac{1}{n-j+1}(z_{j+1} - z_j) \leqslant v_j \leqslant z_{j+1} - \frac{1}{j+1}(z_{j+1} - z_j) \tag{4}$$

$$(j = 1, \dots, n-1).$$

Proof. Since

$$\sum_{i=1}^{n} \frac{1}{v_j - z_i} = \frac{p'(v_j)}{p(v_j)} = 0,$$

we have

$$\sum_{i=1}^{j} \frac{1}{v_j - z_i} = \sum_{k=j+1}^{n} \frac{1}{z_k - v_j}, \tag{5}$$

where the terms in both sums are positive. Owing to this and to the inequality $z_{j+1} - v_j \leqslant z_k - v_j$ for $j+1 \leqslant k$, we obtain

$$\frac{1}{v_j - z_j} \leqslant \frac{n-j}{z_{j+1} - v_j},$$

which implies the left inequality (4). The right inequality (4) can be obtained also from (5) using the inequality $v_j - z_j \leqslant v_j - z_i$ for $i \leqslant j$.

Proof of Lemma 2. We remark firstly that since the sum of the $n+1$ positive numbers $z_1 + 1$, $z_2 - z_1$, ..., $z_n - z_{n-1}$, $1 - z_n$ is 2, one of the following inequalities holds:

$$z_1 + 1 \geqslant \frac{2}{n+1}, \tag{6}$$

$$z_{j+1} - z_j \geqslant \frac{2}{n+1} \quad \text{with} \quad 1 \leqslant j \leqslant n-1, \tag{7}$$

$$1 - z_n \geqslant \frac{2}{n+1}. \tag{8}$$

If (6) holds, then

$$w_0 = |p(v_0)| = |p(-1)| = c \prod_{i=1}^{n} |1 + z_i| \geqslant c(1+z_1)^n \geqslant \frac{2^n c}{(n+1)^n}.$$

In the same way (8) implies

$$w_n \geqslant \frac{2^n c}{(n+1)^n}.$$

If (7) holds, then, by Lemma 3, we have

$$v_j - z_j \geqslant \frac{1}{n}(z_{j+1} - z_j) \geqslant \frac{2}{n(n+1)} \geqslant \frac{2}{(n+1/2)^2},$$

$$z_{j+1} - v_j \geqslant \frac{2}{(n+1/2)^2}.$$

Therefore also $z_i - v_j \geqslant 2/(n+1/2)^2$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n-1$. Hence,

$$w_j \geqslant \frac{2^n c}{(n+1/2)^{2n}}$$

and we have proved that

$$\min_{0 \leqslant j \leqslant n} \frac{1}{w_j} \leqslant \frac{(n+1/2)^{2n}}{2^n c}. \tag{9}$$

By the hypothesis of Lemma 2 one of the following cases holds:

(i) $c \leqslant d$. Then for $-1 \leqslant t \leqslant 1$ we have

$$|p(t)| \leqslant d \prod_{i=1}^{n} |t - z_i| \leqslant 2^n d \text{ and } w_j \leqslant 2^n d \text{ for } j = 0, \ldots, n.$$

(ii) $1/c \leqslant d$. Then (9) implies

$$\min_{0 \leqslant j \leqslant n} \frac{1}{w_j} \leqslant \frac{(n+1/2)^{2n}}{2^n} d.$$

(iii) $z_1 + 1 \leqslant d$. If $c < 1/\sqrt{d}$, then

$$w_0 = |p(v_0)| = c|1 + z_1| \prod_{i=2}^{n} |v_0 - z_i| \leqslant 2^{n-1} cd < 2^{n-1} \sqrt{d}.$$

If $c \geqslant 1/\sqrt{d}$ then (9) implies

$$\min_{0 \leqslant j \leqslant n} \frac{1}{w_j} \leqslant \frac{(n+1/2)^{2n}}{2^n} \sqrt{d}.$$

(iv) $z_{j+1} - z_j \leqslant d$ with $1 \leqslant j \leqslant n-1$. If $c < 1/d$, then

$$w_j = |p(v_j)| \leqslant c(v_j - z_j)(z_{j+1} - v_j) \prod_{\substack{i=1 \\ i \neq j, j+1}}^{n} |z_i - v_j| \leqslant 2^{n-2} cd^2$$

$\leqslant 2^{n-2} d$. If $c \geqslant 1/d$, then (ii) holds.

(v) $1 - z_n \leqslant d$. This case is analogous to (iii).

Therefore, one of the numbers $w_0, \frac{1}{w_0}, w_1, \frac{1}{w_1}, \ldots, w_n, \frac{1}{w_n}$ is not greater than $2^n d$, if (i), $\frac{(n+1/2)^{2n}}{2^n} - d$, if (ii), $2^{n-1}\sqrt{d}$ or $\frac{(n+1/2)^{2n}}{2^n} - \sqrt{d}$, if (iii) or (v), $2^{n-2} d$, if (iv).

This implies the conclusion of Lemma 2.

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MATHEMATICS

On Certain Properties of φ-functions

by

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1. In this paper we use the same definitions and notations as in [3]. Given a φ-function let us define for $\lambda > 0$ the following extended-value functions

$$(\circ)\quad \underline{h}_\varphi(\lambda) = \liminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad (\circ\circ)\quad \overline{h}_\varphi(\lambda) = \limsup_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)},$$

$$(\circ')\quad \underline{h}^a_\varphi(\lambda) = \inf_{u>0} \frac{\varphi(u)}{\varphi(\lambda u)}, \qquad (\circ\circ')\quad \overline{h}^a_\varphi(\lambda) = \sup_{u>0} \frac{\varphi(u)}{\varphi(\lambda u)}.$$

1.1. *There exist limits*

$$(*)\quad s_\varphi = \lim_{\lambda\to 0} \frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda}, \qquad (**)\quad \sigma_\varphi = \lim_{\lambda\to 0} \frac{\lg \overline{h}_\varphi(\lambda)}{-\lg \lambda},$$

and there exist limits (let s^a_φ resp. σ^a_φ stand for these) of analogous expression which appear at (*), (**) on replacing $h_\varphi(\lambda)$, resp. $\overline{h}_\varphi(\lambda)$, by $\underline{h}^a_\varphi(\lambda)$, resp. $\overline{h}^a_\varphi(\lambda)$.

To prove this let us notice that $\underline{h}_\varphi(\lambda) \geqslant 1$ for $0 < \lambda \leqslant 1$,

$$\underline{h}_\varphi(\lambda\mu) = \liminf_{u\to\infty} \frac{\varphi(u)}{\varphi(\lambda u)} \cdot \frac{\varphi(\lambda u)}{\varphi(\lambda\mu u)} \geqslant \underline{h}_\varphi(\lambda)\, \underline{h}_\varphi(\mu), \quad 0 < \lambda, \mu \leqslant 1,$$

and $\overline{h}_\varphi(\lambda\mu) \leqslant \overline{h}_\varphi(\lambda)\, \overline{h}_\varphi(\mu)$. By suitable substitution we reduce the supermultiplicative function $\underline{h}_\varphi(\lambda)$, resp. the submultiplicative function $\overline{h}_\varphi(\lambda)$ to a superadditive, resp. subadditive one and apply the well-known theorem on limits of such functions. For the reader's convenience we shall supply a direct proof in the case (*). Let $0 < \lambda_0 < 1$; given $0 < \lambda < 1$, we determine a positive integer n so that $\lambda = \lambda_0^n \mu$, where $\lambda_0 < \mu \leqslant 1$. $\underline{h}_\varphi(\lambda)$ being supermultiplicative, we obtain $\underline{h}_\varphi(\lambda) = \underline{h}_\varphi(\lambda_0^n \mu) \geqslant (\underline{h}_\varphi(\lambda_0))^n\, \underline{h}_\varphi(\mu)$. Hence,

$$\frac{\lg \underline{h}_\varphi(\lambda)}{-\lg \lambda} \geqslant -\frac{n}{n \lg \lambda_0 + \lg \mu} \lg \underline{h}_\varphi(\lambda_0) + \frac{\lg \underline{h}_\varphi(\mu)}{-\lg \lambda}.$$

If $h_\varphi(\lambda_0)<\infty$, we get with $\lambda\to 0+$

$$\varliminf_{\lambda\to 0+}\frac{\lg h_\varphi(\lambda)}{-\lg\lambda}\geqslant\frac{\lg h_\varphi(\lambda_0)}{-\lg\lambda_0}.$$

The latter inequality holds also in the case of $h_\varphi(\lambda_0)=\infty$ (in that case $h_\varphi(\lambda)=\infty$ for every $\lambda, 0<\lambda<x_0$). Thus we have

$$\varliminf_{\lambda\to 0+}\frac{\lg h_\varphi(\lambda)}{-\lg\lambda}\geqslant\sup_{0<\lambda<1}\frac{\lg h_\varphi(\lambda)}{-\lg\lambda}\geqslant\varlimsup_{\lambda\to 0+}\frac{\lg h_\varphi(\lambda)}{-\lg\lambda},$$

whence

$$(+)\qquad \lim_{\lambda\to 0+}\frac{\lg h_\varphi(\lambda)}{-\lg\lambda}=\sup_{0<\lambda<1}\frac{\lg h_\varphi(\lambda)}{-\lg\lambda}.$$

Similarly we find

$$(++)\qquad \lim_{\lambda\to 0+}\frac{\lg \bar{h}_\varphi(\lambda)}{-\lg\lambda}=\inf_{0<\lambda<1}\frac{\lg \bar{h}_\varphi(\lambda)}{-\lg\lambda}.$$

Relations analogous to $(+)$ and $(++)$ hold for $\underline{h}^a_\varphi(\lambda)$, $\bar{h}^a_\varphi(\lambda)$.

1.2. *If $\varphi\overset{l}{\sim}\psi$ then $s_\varphi=s_\psi$, $\sigma_\varphi=\sigma_\psi$; if $\varphi\overset{a}{\sim}\psi$ then $s^a_\varphi=s^a_\psi$, $\sigma^a_\varphi=\sigma^a_\psi$.*

If $\varphi\overset{l}{\sim}\psi$, we have with certain positive constants a, b, k_1, k_2

$$a\varphi(k_1 u)\leqslant\psi(u)\leqslant b\varphi(k_2 u)$$

for sufficiently large u. Hence, we obtain

$$\frac{a}{b}\,\frac{\varphi(k_1 u)}{\varphi\left(\frac{k_2}{k_1}\lambda k_1 u\right)}=\frac{a}{b}\,\frac{\varphi(k_1 u)}{\varphi(k_2\lambda u)}\leqslant\frac{\psi(u)}{\psi(\lambda u)}\leqslant\frac{b}{a}\,\frac{\varphi(k_2 u)}{\varphi(\lambda k_1 u)}=\frac{b}{a}\,\frac{\varphi(k_2 u)}{\varphi\left(\frac{k_1}{k_2}\lambda k_2 u\right)},$$

$$\frac{a}{b}\,h_\varphi\left(\frac{k_2}{k_1}\lambda\right)\leqslant h_\psi(\lambda)\leqslant\frac{b}{a}\,h_\varphi\left(\frac{k_1}{k_2}\lambda\right),$$

$$\frac{\lg h_\psi(\lambda)}{-\lg\lambda}\leqslant\frac{\lg\frac{b}{a}}{-\lg\lambda}+\frac{\lg h_\varphi\left(\frac{k_1}{k_2}\lambda\right)-\lg\frac{k_1}{k_2}\lambda}{-\lg\frac{k_1}{k_2}\lambda}\cdot\frac{-\lg\frac{k_1}{k_2}\lambda}{-\lg\lambda},\quad\text{for } 0<\lambda<1,$$

whence $s_\psi\geqslant s_\varphi$; we show similarly that $s_\varphi\geqslant s_\psi$. Our argument is untouched in the case a. One can prove similarly $\sigma_\varphi=\sigma_\psi$, resp. $\sigma^a_\varphi=\sigma^a_\psi$.

1.3. (a) *Let $0<s<\infty$; in order that $s_\varphi=s$, it is necessary and sufficient that*

$$(\circ)\qquad \varphi(u)=u^s\varrho(u),$$

where $\varrho(u)$ is continuous, positive for $u>0$ and subject to

$$(+)\qquad \lim_{\lambda\to 0}\frac{\lg\varlimsup_{u\to\infty}\frac{\varrho(u)}{\varrho(\lambda u)}}{-\lg\lambda}=0.$$

(b) *Let* $0<\sigma<\infty$; *in order that* $\sigma_\varphi=\sigma$ *it is necessary and sufficient that*

$$(\circ\circ) \qquad \varphi(u)=u^\sigma\,\varrho_1(u),$$

where $\varrho_1(u)$ *is continuous, positive for* $u>0$ *and subject to*

$$(++) \qquad \lim_{\lambda\to 0+}\frac{\lg\overline{\lim}_{u\to\infty}\dfrac{\varrho_1(\lambda u)}{\varrho_1(u)}}{-\lg\lambda}=0.$$

We shall get analogous theorems for indices s_φ^a, σ_φ^a *if we replace in the above formulae* $\overline{\lim_{u\to\infty}}\dfrac{\varrho(u)}{\varrho(\lambda u)}$ *by* $\inf_{u>0}\dfrac{\varrho(u)}{\varrho(\lambda u)}$ resp. $\overline{\lim_{u>0}}\dfrac{\varrho_1(u)}{\varrho_1(\lambda u)}$ *by* $\sup_{u>0}\dfrac{\varrho_1(u)}{\varrho_1(\lambda u)}$, (on the whole ϱ, ϱ_1 are not φ-functions).

1.4. φ is said to satisfy the *condition* (Δ_α), $\alpha>1$, *for large* u, if $\varphi(\alpha u)\leqslant d_\alpha\varphi(u)$ for $u>u(\alpha)$; φ is said to satisfy the *condition* (Δ_α), $\alpha>1$, *for large* u, if with a $c_\alpha>1$ we have $\varphi(u)c_\alpha\leqslant\varphi(\alpha u)$ for $u>u_1(\alpha)$. If $u(\alpha)$ resp. $u_1(\alpha)$ are zero, then we say that the condition (Δ_α) resp. (Δ_α) is satisfied *for all* u (cf. [3]).

1.4.1. (a) *If* $s_\varphi>0$, *then* (Δ_α) *is satisfied for large* u *and one may pick up, given an arbitrary* $\varepsilon>0$, *some* $\alpha>1$ *and* c_α *so that* $(\lg c_\alpha)(\lg\alpha)^{-1}>s_\varphi-\varepsilon$. *If* (Δ_α) *holds for large* u, *we have* $s_\varphi\geqslant(\lg c_\alpha)(\lg\alpha)^{-1}>0$.

(b) *If* $\sigma_\varphi<\infty$ *then* (Δ_α) *is satisfied for large* u *and one may pick up, given an arbitrary* $\varepsilon>0$, *some* $\alpha>1$, d_α *so that* $(\lg d_\alpha)(\lg\alpha)^{-1}<\sigma_\varphi+\varepsilon$. *If* (Δ_α) *holds for large* u, *we have* $\sigma_\varphi\leqslant(\lg d_\alpha)(\lg\alpha)^{-1}<\infty$.

Analogous theorems hold for indices s_φ^a resp. σ_φ^a *and conditions* (Δ_α) resp. (Δ_α) *for all* u.

Let $s_\varphi<\infty$, $s_\varphi-\varepsilon\,2=s'$, $s_\varphi-\varepsilon=s$ and let with a certain $0<\lambda_0<1$ be

$$\frac{\lg h_\varphi(\lambda_0)}{-\lg\lambda_0}>s',$$

whence $h_\varphi(\lambda_0)\geqslant\lambda_0^{-s'}$ and for $0<c<1$ we have, provided $u\geqslant u_0(c)$, $\varphi(u)\,\varphi(\lambda_0 u)\geqslant c^{s'}\lambda_0^{-s'}$. Let $\alpha=1\,\lambda_0$; for $u>u_0(c)$ we have $\varphi(\alpha u)>c^{s'}\alpha^{s'}\varphi(u)$. It follows that choosing c sufficiently near to 1, $c\alpha>1$, we get for sufficiently large u the inequality $\varphi(\alpha u)\geqslant c_\alpha\varphi(u)$, where $c_\alpha=(c\alpha)^{s'}$,

$$\frac{\lg c_\alpha}{\lg\alpha}=s'+s'\frac{\lg c}{\lg\alpha}>s.$$

For $s_\varphi=\infty$ we choose $s<s'<\infty$ arbitrarily. Suppose φ satisfying (Δ_α) for large u with a certain $\alpha>1$ and with constant c_α. Hence, for large u,

$$\frac{\varphi(u)}{\varphi(\lambda_0 u)}\geqslant c_\alpha,\ \text{with}\ \lambda_0=\frac{1}{\alpha},$$

$h_\varphi(\lambda_0)\geqslant c_\alpha$, $\lg h_\varphi(\lambda_0)-\lg\lambda_0\geqslant\lg c_\alpha/\lg\alpha$ and it remains to appeal to 1.1 (++).

(b) can be proved similarly.

We arrive at the following corollary: $\sigma_\varphi < \infty$ *is equivalent to* (Δ_α), $s_\varphi > 0$ *is equivalent* to (Δ_α).

2. Let ϱ be an arbitrary function continuous and non-vanishing for $u > 0$. ϱ will be said *pseudo-increasing for large* u if there exist positive constants m, n such that

$$\varrho(u_2) \geqslant m\varrho(nu_1) \text{ for } u_2 \geqslant u_1 \geqslant u_0;$$

changing in the above inequality sign $\geqslant$ to $\leqslant$ we come to the definition of *pseudo-decreasing* (*for large* u) function; if $u_0 = 0$ we get the definitions of pseudo-increasing (-decreasing) function *for all* u.

φ-function φ is said to be (Δ, s) resp. (Δ, σ)-representable $(s > 0, \sigma > 0)$ if $\varphi(u) = \psi(u^s)$, where ψ is a convex φ-function, resp. $\varphi(u) = \psi(u^\sigma)$, where ψ is a concave φ-function. We have for (Δ, s)-representable functions $s_\varphi = ss_\psi$, $\sigma_\varphi = s\sigma_\psi$ and analogous relations hold for (Δ, σ)-representable functions.

2.1. (a) *In order that* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, s)-*representable, it is necessary and sufficient that* φ *be of form* 1.3, (o), *where* ϱ *is pseudo--increasing for large* u.

(b) *In order that* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, σ)-*representable, it is necessary and sufficient that* φ *be of form* 1.3, (oo), *where* ϱ_1 *is pseudo--decreasing for large* u.

The theorem does not cease to hold if we replace $\overset{l}{\sim}$ by $\overset{a}{\sim}$ and the words "for large u" by "for all u".

For $s = 1$, resp. $\sigma = 1$, the above is equivalent to theorem 2.3 from [3] and the case of arbitrary s, σ may be switched to the latter one. For the details of proof see [4].

2.2. *If* (+) $\varphi(au) \leqslant c a^s \varphi(u/\varepsilon)$ $(s > 0)$, *where* $c > 0$, *for* $0 \leqslant a \leqslant 1$ *and for* (++) $a^s \varphi(u/\varepsilon) > a > 0$, *then* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, s)-*representable.*

Let $a^s = (\varphi(u/\varepsilon))^{-1} a$; we have $\varphi(a^{1/s} u \varphi(u/\varepsilon)^{-1/s}) \leqslant ca$, whence $\lim_{u\to\infty} u^{-s}\varphi(u\varepsilon) > d > 0$. Let $u_2 > u_1 > u_0$, where $u^{-s}\varphi(u\varepsilon) > d/2$ for $u \geqslant u_0$, $a = u_1/u_2$. Since for sufficiently large u_0 we have $(u_1/u_2)^s \varphi(u_2/\varepsilon) > u_0^s d/2 > a$ it follows that $\varphi(au_2) = \varphi(u_1) \leqslant c(u_1/u_2)^s \varphi(u_2/\varepsilon)$, i.e. writing $\varrho(u) = \varphi(u)u^{-s}$ we get $\varrho(u_1) \leqslant c\varrho(u_2/\varepsilon)$ for large u.

2.2.1. *If* (+) *holds for those* a, u *for which* (++) *is satisfied and, moreover, for* $0 < a < 1$ *and* u *sufficiently small, then* $\varphi \overset{a}{\sim} \chi$, *where* χ *is* (Δ, s)-*representable.*

2.3. (a) *If* $s_\varphi > 0$ *then with every* $s < s_\varphi$ *we have* $\varphi \overset{l}{\sim} \chi_s$, *where* χ_s *is* (Δ, s)-*representable, we have* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, s_φ)-*representable, if and only if* φ *is of form* 1.3, (o) *with* $s = s_\varphi$ *and* ϱ *pseudo-increasing for large* u *and satisfying* 1.3, (+). *With no* $s > s_\varphi$ φ *is l-equivalent to a* (Δ, s)-*representable* χ_s.

(b) *If* $\sigma_\varphi < \infty$, *then with every* $\sigma_\varphi < \sigma$ *we have* $\varphi \overset{l}{\sim} \chi_\sigma$, *where* χ_σ *is* (Δ, σ)*-representable; we have* $\varphi \overset{l}{\sim} \chi$, *where* χ *is* (Δ, σ_φ)*-representable, if and only if* ψ *is of form* 1.3, (oo) *with* $\sigma = \sigma_\varphi$ *and* ϱ_1 *pseudo-decreasing for large* u *and satisfying* 1.3, $(\vdash\vdash)$. *With no* $\sigma < \sigma_\varphi$ φ *is l-equivalent to a* (Δ, σ)*-representable* χ_σ.

The theorem is still valid if we replace in its formulation s_φ, σ_φ *by* s^a_φ, σ^a_φ, $\overset{l}{\sim}$ *by* $\overset{a}{\sim}$, *the words "for large u" by "for all u".*

In the proof of (a) we make use of Theorem 2.2 [3] (for details see [4]) and 1.4.1, (a), 2.1, (a), in the proof of (b) of Theorem 2.1 [3] and 1.4.1, (b), 2.1, (b).

2.4. Now we shall give few examples of φ-functions and the corresponding indices s_φ, σ_φ. If φ is convex, then from the inequality $\varphi(\lambda u) < \lambda \varphi(u)$ for $0 < \lambda < 1$ it follows $s_\varphi \geqslant s^a_\varphi \geqslant 1$; if φ is concave, then $\sigma_\varphi \leqslant \sigma^a_\varphi \leqslant 1$. If there exist

$$\lim_{u \to \infty} \frac{\varphi(u)}{\varphi(\lambda u)} = \underline{h}_\varphi(\lambda) = \overline{h}_\varphi(\lambda) < \infty \text{ for every } \lambda > 0,$$

then $h_\varphi(\lambda\mu) = h_\varphi(\lambda) h_\varphi(\mu)$ for $\lambda, \mu > 0$, whence $\underline{h}_\varphi(\lambda) = 1$ or $\lambda^s (s > 0)$ for $\lambda > 0$; therefore two cases are possible: $s_\varphi = \sigma_\varphi = 0$ or $s_\varphi = \sigma_\varphi = s$. If $\underline{h}_\varphi(\lambda) = \overline{h}_\varphi(\lambda) = 1$ for $\lambda > 0$, then we have the so-called slowly oscillating function in the sense of Karamata [1]. If $\varphi(u) \sim \psi(u) \varrho(u)$, where $s_\psi > 0$, then, owing to 1.2, we get $s_\varphi = \sigma_\varphi = \infty$, since we have always $s_{\varphi\psi} \geqslant s_\varphi + s_\psi$, $\sigma_{\varphi\psi} \leqslant \sigma_\varphi + \sigma_\psi$. The case $\psi(u) = u$ means the functions satisfying the condition (Δ_3), to adopt the notation of [2]. Let $\varphi(u) = u^s \varrho(u)$, $\varphi_1(u) = u^s \varrho(u)$, where $\varrho(u)$ is a slowly oscillating function (e.g. $\varrho(u) = \lg(1+u)$). We obtain $s_\varphi = \sigma_\varphi = s$, $s_{\varphi_1} = \sigma_{\varphi_1}$, but φ is not l-equivalent to a (Δ, s)-representable φ-function and φ_1 is not l-equivalent to a (Δ, s)-representable φ-function. For it may be easily seen that $1/\varrho(u)$ is not pseudo-increasing for large u and $\varrho(u)$ is not pseudo-decreasing for large u.

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MATHÉMATIQUE

Sur une classe de fonctions discontinues et une intégrale singulière dans l'espace

par

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I. Une classe de fonctions discontinues dans l'espace

Dans cette communication nous donnons la définition d'une classe de fonctions discontinues dans l'espace, analogue à la classe $\mathfrak{H}_\alpha^h$ de fonctions discontinues, définies sur un ensemble d'arcs dans le plan de la variable complexe, que nous avons étudié dans nos travaux antérieurs [1], [2]. Ensuite nous étudierons la propriété principale de cette classe relativement à une intégrale dans l'espace. Soit donc dans l'espace euclidien E_n à n dimensions un ensemble fini de $p+1$ surfaces fermées de Liapounoff $S_0\ S_1, ..., S_p$ à $n-1$ dimensions ($p \geqslant 0$). On admet que ces surfaces n'ont pas des points communs et que les surfaces $S_1, ..., S_p$ sont situées d'une façon arbitraire à l'intérieur du domaine borné Ω_0 limité par la surface S_0. Désignons par Ω un ensemble de tous les points du domaine Ω_0 non situés sur les surfaces $S_1, S_2, ..., S_p$. Cet ensemble est une somme $\Omega = \Omega_1 + \Omega_2 + \dots + \Omega_m$ de domaines disjoints $\Omega_1, ..., \Omega_m$ simplement ou multiplement connexes. Dans le cas de l'absence des surfaces $S_1, ..., S_p$ on a $\Omega = \Omega_0$.

DÉFINITION. Nous appelons classe $\mathfrak{H}_\alpha^h$ un ensemble de toutes les fonctions complexes ou réelles $f(x)$, définies dans l'ensemble de points $x(\xi_1, ..., \xi_n) \in \Omega$ (ou, peut être, dans $\Omega_0 + S_0$), qui vérifient inégalité

$$|f(x)| < M_f |x - x_S|^{-\alpha}, \qquad (x \in \Omega) \tag{1}$$

et une condition généralisée de Hölder

$$|f(x) - f(y)| < \frac{k_f |x-y|^h}{|x-x_S|^{\alpha+h}}, \tag{2}$$

où $|x-y|$ désigne la distance euclidienne entre deux points arbitraires $x(\xi_1, ..., \xi_n)$ et $y(\eta_1, ..., \eta_n)$ situés simultanément à l'intérieur du même domaine $\Omega_1, ..., \Omega_m$; x_S désigne un point des surfaces de discontinuité $S_0, S_1, ..., S_p$, où la distance $|x - x_S|$ du point intérieur $x \in \Omega$ atteint sa borne in-

férieure; on admet que $|x-x_S| \leqslant |y-y_S|$; les paramètres constants a, h de la classe vérifient les inégalités

$$0 \leqslant a < 1; \quad 0 < h < 1; \quad a+h < 1; \tag{3}$$

les constantes positives M_f et k_f sont arbitraires.

THÉORÈME 1. *La classe $\mathfrak{H}^{h_1}_{a_1}$ fait partie de la classe $\mathfrak{H}^{h}_{a}$ si les paramètres de ces classes vérifient les inégalités*

$$a_1 \leqslant a; \quad h_1 \geqslant h.$$

La démonstration est analogue à celle donnée dans [1].

II. Une intégrale singulière dans l'espace

Les intégrales singulières dans l'espace étaient étudiées par Tricomi [3], Giraud [4], Michline [5] et récemment par A. Zygmund [6]. Nous adopterons la forme suivante de l'intégrale singulière, empruntée au dernier auteur:

$$\int_{\Omega} N(x-y) f(y)\, dy = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} N(x-y) f(y)\, dy, \tag{4}$$

où Ω_ε désigne un ensemble de tous les points y de l'ensemble Ω dont la distance $|x-y|$ du point intérieur x est supérieure au nombre positif ε, $f(y)$ est une fonction complexe intégrable, définie dans Ω et $N(x)$ est une fonction de la forme

$$N(x) = K(x')\,|x|^{-n}; \quad (|x| \neq 0), \tag{5}$$

où x' est la projection centrale du point x sur la surface d'une sphère ω, centrée au point $(0, 0, \ldots, 0)$, de rayon unité, on a donc $x = x'|x|$. On suppose que la fonction complexe $K(x')$, définie sur la surface sphérique ω, vérifie la condition fondamentale

$$\int_{\omega} K(x')\, dx' = 0. \tag{6}$$

On suppose en outre que la fonction $K(x')$ vérifie sur la surface ω une condition simple de Hölder

$$|K(x') - K(y')| < k_\omega |x' - y'|^{h_\omega}, \qquad (0 < h_\omega \leqslant 1). \tag{7}$$

La propriété principale des fonctions de classe $\mathfrak{H}^{h}_{a}$ dans l'espace, qui est une extension de la propriété analogue des fonctions de classe $\mathfrak{H}^{h}_{a}$ sur un système d'arc démontrée dans [2], est exprimée dans le

THÉORÈME 2. *Si une fonction complexe $f(y)$, définie dans l'ensemble Ω, appartient à la classe $\mathfrak{H}^{h}_{a}$, alors la fonction $\varphi(x)$, définie par intégrale singulière*

$$\varphi(x) = \int_{\Omega} N(x-y) f(y)\, dy \tag{8}$$

en tout point $x \in \Omega$, appartient à la classe $\mathfrak{H}_\alpha^{h'}$, notamment elle vérifie les inégalités

$$|\varphi(x)| < (C_1 M_f + C_2 k_f)|x - x_S|^{-\alpha} \tag{9}$$

$$|\varphi(x) - \varphi(z)| < (C_1' M_f + C_2' k_f |x - x_S|^{-\alpha - h'} |x - z|^{h'} \tag{9'}$$

où $h' = \min(h, h_\omega)$, $(h \neq h_\omega)$ M_f et k_f sont des constantes positives caractérisant la fonction $f(x)$ dans les inégalités (1) et (2), C_1, C_2, C_1', C_2' sont des constantes positives, indépendantes de la fonction f, on a en outre $|x - x_S| \leqslant |z - z_S|$, $x, z \in \Omega_\nu$ $(\nu = 1, 1, ..., m)$

DÉMONSTRATION. L'intégrale (8) étant une somme des intégrales étendues aux domaines $\Omega_1, \Omega_2, ..., \Omega_m$, il suffit d'étudier l'une d'elles

$$J(x) = \int_{\Omega_\nu} N(x - y) f(y)\, dy. \tag{10}$$

Étudions d'abord le cas où point x est situé à l'intérieur du domaine d'intégration Ω_ν. Considérons dans ce but une sphère Π centrée en x et tangente au point le plus rapproché x_S à une des surfaces S_ν. qui limitent le domaine Ω_ν. L'existence de l'intégrale (10) en tout point $x \in \Omega_\nu$ résulte de l'existence de l'intégrale singulière

$$J(x) = \int_{\Pi} N(x - y) f(y)\, dy = \lim_{\varepsilon \to 0} \left\{ \int_{\Pi_\varepsilon} N(x - y)[f(y) - f(x)]\, dy + \right. \tag{11}$$

$$\left. + f(x) \int_{\Pi_\varepsilon} N(x - y)\, dy \right\} = \int_{\Pi} N(x - y)[f(y) - f(x)]\, dy.$$

On arrive à cette conclusion en remarquant que: 1° l'intégrale de la fonction $N(x - y)$ étendue à la portion Π_ε $(|x - y| > \varepsilon)$ de la sphère Π est nulle, en vertu de la supposition (6), 2° la dernière intégrale dans l'égalité (11) admet une faible singularité si $|x - y| \to 0$. Pour trouver une limitation de l'intégrale (11), appliquons l'inégalité admise (2) et remarquons que la distance $|y - y_\Pi|$ du point y de la surface de la sphère Π est inférieure aux distances $|x - x_S|$ et $|y - y_S|$. Nous aurons alors

$$|J^\Pi(x)| < \text{const.} \int_{\Pi} \frac{k_f\, dy}{|x - y|^{n-h} |y - y_\Pi|^{\alpha+h}} = \tag{12}$$

$$= \text{const.}\, k_f \int_0^{|x - x_S|} \frac{\varrho^{n-1}\, d\varrho}{\varrho^{n-h}[|x - x_S| - \varrho]^{\alpha+h}} < \frac{\text{const.}\, k_f}{|x - x_S|^\alpha}.$$

Pour l'intégrale étendue à la portion extérieure $\Omega_\nu - \Pi$, dont l'existence est assurée en vue de la supposition (1), nous aurons une inégalité

$$|J^{\Omega_\nu - \Pi}(x)| < \text{const.} \int_{\Omega_\nu - \Pi} \frac{M_f\, dy}{|x - y|^n\, y - y_S|^\alpha}. \tag{13}$$

Nous allons étudier cette intégrale d'une manière analogique à celle employée dans [7]. Il suffit notamment d'étudier la partie de la dernière intégrale étendue au domaine $\widetilde{W}$, commun au domaine $\Omega_\nu - \Pi$ et au cylindre W (centré en x_S) de l'axe normale en x_S, de l'hauteur $2r_W$ et de rayon r_W suffisamment petit pour que la condition connue de Liapounoff soit applicable à la portion de la surface S_ν, située à l'intérieur de W. On peut admettre que $4|x-x_S|$ est inférieure à l'hauteur du cylindre W. En se servant d'une transformation homothétique du point x_S comme centre dans le rapport $1/|x-x_S|$, on aura [7]

$$(14)\qquad \int\limits_{\widetilde{W}} \frac{dy}{|x-y|^n |y-y_S|^\alpha} = \frac{1}{|x-x_S|^\alpha} \int\limits_{\widetilde{W}'} \frac{dy'}{|x'-y'|^n |y'-y'_S|^\alpha} <$$

$$< \frac{1}{|x-x_S|^\alpha} \left[\int\limits_{\widetilde{W}'_\Pi} \frac{dy'}{|x'-y'|^n |y-y'_S|^\alpha} + \text{const.} \int\limits_1^{r_W/|x-x_S|} \varrho^{n-2} d\varrho \int\limits_0^\infty \frac{d\xi}{\sqrt{(\varrho^2+\xi^2)^n}\, \xi^\alpha} \right],$$

$\widetilde{W}'_\Pi$ désignant le domaine transformé du domaine $\widetilde{W}_\Pi$ commun au domaine $\widetilde{W}$ et au cylindre, centré en x_S, de rayon $|x-x_S|$ et de la hauteur $4|x-x_S|$. Or, les intégrales obtenues, grace à l'inégalité (14), sont bornées, si $|x-x_S|$ tend vers zéro. Il en résulte, en tenant compte de (12), la limitation cherchée

$$(15)\qquad |J(x)| < (c_1 M_f + c_2 k_f) |x-x_S|^{-\alpha}$$

de l'intégrale (10), si $x \in \Omega_\nu$. Dans le cas, où le point x est situé à l'extérieur du domaine Ω_ν, suffisamment près de la surface S_ν, nous aurons directement

$$(16)\qquad |J(x)| < \text{const.}\, M_f \int\limits_{\Omega_\nu} \frac{dy}{|x-y|^n |y-y_S|^\alpha}.$$

Cette intégrale sera étudiée d'une manière analogique à celle de l'étude de l'integrale (14) [7] et on arrivera à une limitation de la forme (15), sans le terme en k_f. En somme, nous arrivons à la conclusion de la première partie (9) du Théorème 2. Pour démontrer la propriété (9′) de l'intégrale (8), il suffit d'étudier la différence de valeurs de l'intégrale (10) dans la cas $|x-z| \leqslant 1/4 |x-x_S|$, puisque dans le cas contraire la propriété (9′) résulte immédiatement de l'inégalité démontrée (9). Supposons d'abord que les points x et z sont situés dans le domaine d'intégration Ω_ν et décomposons les valeurs de l'intégrale (10) en sommes

$$(17)\qquad \begin{aligned} J(x) &= J^{\Pi_1}(x) + J^{\Pi-\Pi_1}(x) + J^{\Omega_\nu-\Pi}(x), \\ J(z) &= J^{\Pi_2}(z) + J^{\Pi-\Pi_2}(z) + J^{\Omega_\nu-\Pi}(z) \end{aligned}$$

des intégrales étendues aux domaines Π_1, $\Pi - \Pi_1$, $\Omega_\nu - \Pi$, etc., où Π_1 désigne une sphère, centrée en x, de rayon $2|x-z|$ et Π_2-une sphère, centrée en z, de rayon $3|x-z|$. Nous aurons

$$(18)\quad |J^{\Pi_1}(x)| =$$

$$= \left| \int_{\Pi_1} N(x-y)[f(y)-f(x)]\,dy \right| < \text{const.}\, k_f \int_0^{2|x-z|} \frac{\varrho^{n-1}d\varrho}{\varrho^{n-h}(|x-x_S|-\varrho)^{\alpha+h}} <$$

$$< \frac{\text{const.}\, k_f}{|x-x_S|^{\alpha}} \int_0^{2|x-z|:|x-x_S|} \frac{ds}{s^{1-h}(1-s)^{\alpha+h}} < \frac{\text{const.}\, k_f |x-z|^h}{|x-x_S|^{\alpha+h}}.$$

et une limitation analogue pour la valeur $J^{\Pi_2}(z)$. Nous pouvons écrire ensuite, en vertu de (6),

$$(19)\quad J^{\Pi-\Pi_1}(x) = \int_{\Pi-\Pi_1} N(x-y)[f(y)-f(x)]\,dy =$$

$$= \left[\int_{\Pi-\Pi_2'} + \int_{\Pi_2'-\Pi_2} + \int_{\Pi_2-\Pi_1} \right] N(x-y)[f(y)-f(x)]\,dy;$$

$$(19')\quad J^{\Pi-\Pi_2}(z) = \int_{\Pi-\Pi_2} N(z-y)f(y)\,dy = \int_{\Pi-\Pi_2'} N(z-y)f(y)\,dy +$$

$$+ \int_{\Pi_2'-\Pi_2} N(z-y)[f(y)-f(x)]\,dy.$$

Π_2' étant une sphère, centrée en z, de rayon $|x-x_S|-|x-z|$. Il en résulte

$$(20)\quad J^{\Pi-\Pi_1}(x) - J^{\Pi-\Pi_2}(z) = \left[\int_{\Pi-\Pi_2'} + \int_{\Pi_2-\Pi_1} \right] N(x-y)[f(y)-f(x)]\,dy +$$

$$- \int_{\Pi-\Pi_2'} N(z-y)f(y)\,dy + \int_{\Pi_2'-\Pi_2} [N(x-y)-N(z-y)][f(y)-f(x)]\,dy$$

En vertu de (1), (2), (5), nous déduirons les limitations suivantes

$$(21)\quad \left| \int_{\Pi-\Pi_2'} N(x-y)f(y)\,dy \right| < \int_{|x-x_S|-2|x-z|}^{x-x_S} \frac{\text{const.}\, \varrho^{n-1}d\varrho}{\varrho^n(|x-x_S|-\varrho)^{\alpha}} <$$

$$< \frac{\text{const.}}{|x-x_S|^{\alpha}} \int_{1-2|x-z|:|x-x_S|}^{1} \frac{dt}{t(1-t)^{\alpha}} < \frac{\text{const.}\,|x-z|^h}{|x-x_S|^{\alpha-h}}$$

$$(22)\quad \left| \int_{\Pi-\Pi_2'} N(x-y)\,dy \right| < \text{const.} \left| \log\left(1-2\frac{|x-z|}{|x-x_S|}\right) \right| < \text{const.} \frac{|x-z|}{|x-x_S|}.$$

Ensuite, d'après (7) et les inégalités $1/2 < |x-y|/|z-y| < 2$, si $y \in \Pi - \Pi_1$, on trouve par un calcul élémentaire

$$(23) \qquad |N(x-y)-N(z-y)| = \left|\frac{K[(x-y)']}{|x-y|^n} - \frac{K[(z-y)']}{|z-y|^n}\right| < \\ < \text{const.}\left[\frac{|x-z|}{|x-y|^{n+1}} + \frac{\theta^{h_\omega}}{|x-y|^n}\right],$$

θ désignant l'angle entre les vecteurs $x-y$ et $z-y$, pour lequel $\max \theta = \arcsin |x-z|/|x-y|$ si les distances $|x-y|$ et $|x-z|$ sont constantes. Nous en concluons pour la dernière, D, des trois intégrales dans la somme (20) une limitation

$$(24) \qquad |D < \text{const.}\, k_f |x-z|^{h_\omega} \int\limits_{\Pi_2'-\Pi_2} \frac{dy}{|x-y|^{n-h_\omega-h}\,[\,|x-x_S|-|x-y|\,]^{\alpha+h}} < \\ < \frac{\text{const.}\, k_f |x-z|^{h_\omega}}{|x-x_S|^{\alpha+h_\omega}} \int\limits_{\frac{\lambda|x-z|}{|x-x_S|}}^{1} \frac{dt}{t^{1-h-h_\omega}(1-t)^{\alpha+h}} < \frac{\text{const.}\, k_f |x-z|^{h'}}{|x-x_S|^{\alpha+h'}}.$$

Pour la différence des troisièmes intégrales (17), on aura simplement, d'après (23), en répétant le même raisonnement que pour (13), une limitation

$$(25) \qquad |J^{\Omega_\nu - \Pi}(x) - J^{\Omega_\nu - \Pi}(z)| < \text{const.}\, M_f |x-z|^{h_\omega} \int\limits_{\Omega_\nu - \Pi} \frac{dy}{|x-y|^{n+h_\omega}|y-y_S|^\alpha} < \\ < \text{const.}\, M_f |x-x_S|^{-\alpha-h_\omega}|x-z|^{h_\omega}.$$

En rapprochant les résultats (18), (21), (22), (24), (25), on arrive à la propriété cherchée de l'intégrale (10)

$$(26) \qquad |J(x)-J(z)| < (c_1' M_f + c_2' k_f)|x-x_S|^{-\alpha-h'}|x-z|^{h'},$$

si $x, z \in \Omega_\nu$. Dans le cas, où les points x, z sont situés à l'extérieur du domaine Ω_ν, suffisamment rapproché de la surface S_ν, on trouve, par un calcul tout semblable à celui pour la différence (25), une limitation

$$(27) \qquad |J(x)-J(z)| < \text{const.}\, M_f |x-z|^{h_\omega} \int\limits_{\Omega_\nu} \frac{dy}{|x-y|^{n+h_\omega}|y-y_S|^\alpha} < \\ < \text{const.}\, M_f |x-x_S|^{-\alpha-h_\omega}|x-z|^{h_\omega}.$$

En somme, nous déduirons la conclusion du thèse (9′) de notre théorème. Dans le cas $h < h_\omega$ la fonction transformée $\varphi(x)$ appartient à la même classe $\mathfrak{H}_\alpha^h$ que la fonction donnée $f(x)$.

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MATHEMATICS

A Theorem of Toeplitz Type for the Class of M-summable Sequences

by
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Presented by W. ORLICZ on May 12, 1960

The main theorem of this paper, concerning the consistency of summability methods, is given in par. 3. This statement is an analogue of another author's result for integrals, [2].

1. Terminology and introductory remarks

We shall denote by M and N the convex functions complementary in the Young sense, and by M^{-1} and N^{-1} their inverse functions ([1], pp. 16—20).

For any sequence $u = \{u_k\}$ let $\left\|\begin{smallmatrix}n_2\\n_1\end{smallmatrix} u\right\|_M$ be defined as $\sup_v \sum_{k=n_1}^{n_2} |u_k| v_k$, where the supremum is taken over all non-negative sequences $v = \{v_k\}$ for which $\sum_{k=n_1}^{n_2} N(v_k) \leqslant 1$ $(n_1 \leqslant n_2)$.

It is easy to verify that, if $n_1 \leqslant n_2 < n_3$, then

$$\left\|\begin{smallmatrix}n_2\\n_1\end{smallmatrix} u\right\|_M \leqslant \left\|\begin{smallmatrix}n_2\\n_1\end{smallmatrix} u\right\|_M + \left\|\begin{smallmatrix}n_3\\n_2+1\end{smallmatrix} u\right\|_M \quad \text{and} \quad \left\|\begin{smallmatrix}n_2\\n_1\end{smallmatrix} u\right\|_M \leqslant \left\|\begin{smallmatrix}n_3\\n_1\end{smallmatrix} u\right\|_M.$$

From the Young inequality ([1], pp. 24, 89—91) it follows at once that for arbitrary sequences $u = \{u_k\}$ and $w = \{w_k\}$ the analogue of the Hölder inequality

$$\sum_{k=n_1}^{n_2} |u_k w_k| \leqslant \left\|\begin{smallmatrix}n_2\\n_1\end{smallmatrix} u\right\|_M \left\|\begin{smallmatrix}n_2\\n_1\end{smallmatrix} w\right\|_N \qquad (n_1 \leqslant n_2) \tag{1}$$

is true.

Given a characteristic sequence $\chi = \{\chi_k\}$ of the set of positive integers $p_1 < p_2 < \ldots < p_n$ (i. e. $\chi_k = 1$ if $k = p_i$, $i = 1, 2, \ldots, n$, $\chi_k = 0$, otherwise), we have

$$\left\|\begin{smallmatrix}p_n\\p_1\end{smallmatrix} \chi\right\|_N = nM^{-1}\left(\frac{1}{n}\right) \qquad (n \geqslant 1). \tag{2}$$

From (1) and (2) follows the inequality

$$\frac{1}{n}\sum_{k=1}^{n}|u_k| \leqslant M^{-1}\left(\frac{1}{n}\right)\left\|{}_1^n u\right\|_M \qquad (n \geqslant 1). \tag{3}$$

The class of sequences $u = \{u_k\}$ such that

$$\lim_{n\to\infty} M^{-1}\left(\frac{1}{n}\right)\left\|{}_1^n u\right\|_M = 0$$

will be denoted by R_M.

We say that the sequence $\{a_k\}$ is M-summable to g and we write $M\text{-}\lim_{k\to\infty} a_k = g$, if $\{a_k - g\} \in R_M$. The sequence $\{a_k\}$ is called $|C^\alpha, 1|$-summable to g and written $|C^\alpha, 1|\text{-}\lim_{k\to\infty} a_k = g$, if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_k - g|^\alpha = 0$ $(\alpha \geqslant 1)$.

Obviously, $|C^\alpha, 1|\text{-}\lim a_k = M\text{-}\lim a_k$ when $M(x) = x^\alpha$ $(\alpha \geqslant 1)$.

The limit of M-summable sequences is uniquely defined. Every convergent sequence is M-summable (to the same limit).

It is evident from (3) that any M-summable sequence is $|C^1, 1|$-summable. There exist $|C^1, 1|$-summable sequences which are not $|C^\alpha, 1|$-summable with any $\alpha > 1$, and are M-summable with a certain M (it is sufficient to take the sequence $\{a_k\}$, where $a_k = 2^p p^{-2}$, if $k = 2^p$, $p = 1, 2, \ldots$, $a_k = 0$ otherwise, and the function M for which $M^{-1}(x) = x|\ln x|$, if $0 < x < 3^{-1}$).

2. Fundamental lemmas

First we shall prove the following

LEMMA 1. *Let*

$$\beta_k = \frac{1}{M(2^{1-k}M^{-1}(1))}, \quad \gamma_k = [\beta_k], \quad b_k = \frac{1}{M^{-1}\left(\frac{1}{\beta_k}\right)}, \quad c_k = \frac{1}{M^{-1}\left(\frac{1}{\gamma_k}\right)} \quad (k = 1, 2, \ldots).$$

Then

$$\sup_{n\geqslant 1} \frac{1}{c_n}\sum_{k=1}^{n+1} c_k < \infty. \tag{4}$$

Proof. It is easy to observe that $\beta_1 = 1$, $\beta_{k+1} > 2\beta_k$ for $k = 1, 2, \ldots$ ([1], p. 18), $\gamma_1 = 1$, $\gamma_{k+1} > \gamma_k$, $\beta_k \geqslant \gamma_k$ $(k = 1, 2, \ldots)$, $\lim_{k\to\infty} \beta_k = \lim_{k\to\infty} \gamma_k = \infty$, $\gamma_k \simeq \beta_k$ (as $k \to \infty$).

From inequalities

$$\frac{\gamma_k}{\beta_k} \leqslant \frac{M^{-1}\left(\frac{1}{\beta_k}\right)}{M^{-1}\left(\frac{1}{\gamma_k}\right)} < 1$$

the asymptotic equalities

$$c_k \simeq b_k \quad \text{and} \quad \frac{c_{k+1}}{c_k} \simeq \frac{b_{k+1}}{b_k} \quad (\text{as } k \to \infty)$$

follow.

Since $\{b_k\}$ is the geometric sequence with the quotient 2, thus for $k > k_0$, $\frac{3}{2} < \frac{c_{k+1}}{c_k} < \frac{5}{2}$, and for $n > k_0$

$$\frac{1}{c_n} \sum_{k=1}^{n+1} c_k = \frac{1}{c_n} \left(\sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{n+1} \right) c_k \leqslant \frac{1}{c_1} \sum_{k=1}^{k_0} c_k + \frac{15}{2},$$

whence we obtain (4).

Remark. In the case $M(x) = \frac{x^\alpha}{\alpha}$ $(\alpha > 1)$:

$$\beta_k = \gamma_k = 2^{k-1}, \quad b_k = c_k = \left(\frac{2^{k-1}}{\alpha} \right)^{1/\alpha}.$$

Now we present two further lemmas which shall be directly applied in par. 3.

LEMMA 2. *Let γ_k and c_k have the same meaning as in Lemma 1 and let*

$$A_l = \sum_{k=l}^{\infty} c_{k+1} \left\| {}_{\gamma_k}^{\gamma_{k+1}-1} w \right\|_N < \infty$$

with some sequence $w = \{w_k\}$. Then, for every sequence $u = \{u_k\} \in R_M$,

$$B_l(u) = \sum_{k=\gamma_l}^{\infty} |u_k w_k| \leqslant A_l Q(u), \tag{5}$$

where

$$Q_l(u) = \sup_{n \geqslant \gamma_l} M^{-1}\left(\frac{1}{n}\right) \left\| {}_{\gamma_l}^{n} u \right\|_M.$$

Proof. In view of (1)

$$B_l(u) = \sum_{k=l}^{\infty} \sum_{i=\gamma_k}^{\gamma_{k+1}-1} |u_i w_i| \leqslant \sum_{k=l}^{\infty} \left\| {}_{\gamma_k}^{\gamma_{k+1}-1} u \right\|_M \left\| {}_{\gamma_k}^{\gamma_{k+1}-1} w \right\|_N.$$

But

$$\left\| {}_{\gamma_k}^{\gamma_{k+1}-1} u \right\|_M \leqslant \left\| {}_{\gamma_l}^{\gamma_{k+1}-1} u \right\|_M \leqslant \frac{1}{M^{-1}\left(\frac{1}{\gamma_{k+1}}\right)} Q_l(u),$$

hence

$$B_l(u) \leqslant Q_l(u) \sum_{k=l}^{\infty} c_{k+1} \left\| {}_{\gamma_k}^{\gamma_{k+1}-1} w \right\|_N = A_l Q_l(u).$$

In the sequel we shall denote by R_M^* the Banach space of all sequences $u = \{u_k\} \in R_M$ with the norm

$$\|u\| = \sup_{n \geqslant 1} M^{-1}\left(\frac{1}{n}\right) \left\|\sum_1^n u\right\|_M.$$

LEMMA 3. *Let the sequence $w = \{w_k\}$ be such that for any $u = \{u_k\} \in R_M^*$ the series*

$$S(u) = \sum_{k=1}^{\infty} u_k w_k$$

converges. Then S is a linear functional over R_M^* with the norm $\|S\| = \sup_u |S(u)|$ $(\|u\| \leqslant 1)$ satisfying the inequality

$$D \sum_{k=1}^{\infty} c_{k+1} \left\|\sum_{\gamma_k}^{\gamma_{k+1}-1} w\right\|_N \leqslant \|S\| \leqslant \sum_{k=1}^{\infty} c_{k+1} \left\|\sum_{\gamma_k}^{\gamma_{k+1}-1} w\right\|_N, \tag{6}$$

where γ_k and c_k are as defined in Lemma 1, and D is a positive constant.

Proof. 1° Let $S_n(u) = \sum_{k=1}^{n} u_k w_k$. Evidently, S_n is a linear functional over R_M^*, hence the functional S defined as limit $S(u) = \lim_{n \to \infty} S_n(u)$ is linear over R_M^*.

2° The second inequality (6) is a special case of (5). We shall estimate them norm of S from below.

Proceeding as in the case of functionals defined by an integral [1], p. 146) we shall observe at once that for every positive integer ν there is a sequence $u^{(\nu)} = \{u_k^{(\nu)}\}$ such that

$$\left\|\sum_{\gamma_\nu}^{\gamma_{\nu+1}-1} u^{(\nu)}\right\|_M = 1 \quad \text{and} \quad \sum_{k=\gamma_\nu}^{\gamma_{\nu+1}-1} |u_k^{(\nu)} w_k| \geqslant \frac{1}{4} \left\|\sum_{\gamma_\nu}^{\gamma_{\nu+1}-1} w\right\|_N.$$

Consider the sequence $\tilde{u} = \{\tilde{u}_k\}$, where

$$\tilde{u}_k = \begin{cases} c_{\nu+1} |u_k^{(\nu)}| \operatorname{sign} w_k & \text{for } \gamma_\nu \leqslant k < \gamma_{\nu+1},\ \nu = 1, 2, \ldots, p, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\tilde{u} \in R_M^*$. Let us choose, for given n, an integer r such that $\gamma_r \leqslant n < \gamma_{r+1}$.

If $r \leqslant p$, then

$$M^{-1}\left(\frac{1}{n}\right) \left\|\sum_1^n \tilde{u}\right\|_M \leqslant M^{-1}\left(\frac{1}{\gamma_r}\right) \left\|\sum_1^{\gamma_{r+1}-1} \tilde{u}\right\|_M = \frac{1}{c_r} \sup_v \sum_{\nu=1}^{r} c_{\nu+1} \sum_{k=\gamma_\nu}^{\gamma_{\nu+1}-1} |u_k^{(\nu)}| v_k \leqslant$$

$$\leqslant \frac{1}{c_r} \sum_{\nu=1}^{r} c_{\nu+1} \left\|\sum_{\gamma_\nu}^{\gamma_{\nu+1}-1} u^{(\nu)}\right\|_M = \frac{1}{c_r} \sum_{\nu=1}^{r} c_{\nu+1} \leqslant L,$$

where the supremum was taken over all non-negative sequences $v=\{v_k\}$ such that $\sum_{k=1}^{\gamma_{r+1}-1} N(v_k)\leqslant 1$, and L denotes a constant independent of r, which exists by Lemma 1.

If $r>p$, then, just as above,

$$M^{-1}\left(\frac{1}{n}\right)\left\|\sum_1^n \tilde{u}\right\|_M \leqslant M^{-1}\left(\frac{1}{\gamma_{p+1}}\right)\left\|\sum_1^{\gamma_{p+1}-1}\tilde{u}\right\|_M \leqslant \frac{1}{c_{p+1}}\sum_{\nu=1}^{p+1} c_{\nu+1}\leqslant L.$$

Further,

$$S(\tilde{u})=\sum_{\nu=1}^{\infty}\sum_{i=\gamma_\nu}^{\gamma_{\nu+1}-1}\tilde{u}_i w_i=\sum_{\nu=1}^{p} c_{\nu+1}\sum_{i=\gamma_\nu}^{\gamma_{\nu+1}-1}|u_i^{(\nu)}w_i|\geqslant\frac{1}{4}\sum_{\nu=1}^{p} c_{\nu+1}\left\|\sum_{\gamma_\nu}^{\gamma_{\nu+1}-1} w\right\|_N$$

and

$$L\|S\|\geqslant|S(\tilde{u})|, \quad \text{hence} \quad \|S\|\geqslant\frac{1}{4L}\sum_{\nu=1}^{p} c_{\nu+1}\left\|\sum_{\gamma_\nu}^{\gamma_{\nu+1}-1} w\right\|_N,$$

and p being arbitrary, we obtain the first inequality (6).

3. A theorem of Toeplitz type

We shall now give the main

THEOREM 1. *The relation*

$$M\text{-}\lim_{k\to\infty} a_k = g$$

implies

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_k b_k^{(n)} = g \tag{7}$$

for every M-summable sequence $\{a_k\}$ if and only if

$$\lim_{n\to\infty}\sum_{k=r}^{\infty} b_k^{(n)} = 1 \quad \textit{with} \quad r=1,2,\ldots, \tag{8}$$

and

$$\overline{\lim_{n\to\infty}}\sum_{k=1}^{\infty}\frac{1}{M^{-1}\left(\frac{1}{\gamma_{k+1}}\right)}\left\|\sum_{\gamma_k}^{\gamma_{k+1}-1} b^{(n)}\right\|_N<\infty, \tag{9}$$

where $b^{(n)}=\{b_k^{(n)}\}$, and $\{\gamma_k\}$ is defined in Lemma 1.

Proof. The necessity of (8) is evident (we take $\{a_k\}$ such that $a_k=0$ for $k<r$, $a_k=1$ for $k\geqslant r$). By Lemma 3 and the Banach-Steinhaus theorem, we obtain the necessity of (9).

Writing

$$\sum_{k=1}^{\infty} a_k b_k^{(n)}=\left(\sum_{k=1}^{\gamma_l-1}+\sum_{k=\gamma_l}^{\infty}\right)(a_k-g)\,b_k^{(n)}+g\sum_{k=1}^{\infty} b_k^{(n)}$$

and applying Lemma 2 the sufficiency of (8) and (9) may be easily shown.

In particular, when $M(x) = x^{\alpha}/\alpha\,(\alpha>1)$, we have (see Remark in par. 2)

THEOREM 2. *The relation* $C^{\alpha}, 1 - \lim_{k\to\infty} a_k = g$ $(\alpha > 1)$ *implies* (7) *for any* $|C^{\alpha}, 1|$*-summable sequence* $\{a_k\}$ *if and only if the condition* (8) *is satisfied and*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} 2^{k/\alpha} \left(\sum_{i=2^{k-1}}^{2^k-1} b_i^{(n)\beta} \right)^{1/\beta} < \infty,$$

where $1/\alpha + 1/\beta = 1$.

The last theorem corresponds to a theorem of Tandori for singular integrals ([3], pp. 226—228).

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MATHEMATICS

Concerning the Notion of $\mathcal{R}$-Neighbours

by

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I have introduced in [1] some notions concerning the classification of topological spaces from the point of view of the theory of retracts. In particular,

$$X \underset{\mathcal{R}}{\leqslant} Y$$

means that X is homeomorphic to a retract of Y;

$$X \underset{\mathcal{R}}{=} Y$$

means that $X \underset{\mathcal{R}}{\leqslant} Y$ and $Y \underset{\mathcal{R}}{\leqslant} X$, and

$$X \underset{\mathcal{R}}{<} Y$$

means that $X \underset{\mathcal{R}}{\leqslant} Y$, but the relation $Y \underset{\mathcal{R}}{\leqslant} X$ does not hold.

If $X \underset{\mathcal{R}}{<} Y$ and if a space Z satisfying the condition

$$X \underset{\mathcal{R}}{<} Z \underset{\mathcal{R}}{<} Y$$

does not exist, then X is said to be an *$\mathcal{R}$ neighbour of Y on the left*, and Y an *$\mathcal{R}$-neighbour of X on the right.*

The aim of the present note is to give some remarks concerning the dimension of two $\mathcal{R}$-neighbours.

THEOREM 1. *For every n natural, or $n = \aleph_0$ there exists a continuum Y of dimension n which is an $\mathcal{R}$-neighbour on the right for the segment X.*

Proof. Let K be a 1-dimensional continuum which does not contain any simple arc [2] and let K^n denote the Cartesian product of n factors equal to K. Evidently, K^n is an n-dimensional continuum which does not contain any simple arc. We may assume that K^n lies in the subset H_0 of the Hilbert space H consisting of all points $\{x_1, x_2, ..., x_\nu, ...\}$ with $x_1 = 0$. Let us pick up in K^n a fixed point a_0. Since K^n is a continuum, there exists for every natural m, a finite sequence of points

$$a_{m,1}, a_{m,2}, ..., a_{m,j_m} \in K^n$$

satisfying the following conditions:

1° $a_{m,1} = a_0$,

2° $\varrho(a_{m,\nu}, a_{m,\nu+1}) < \frac{1}{m}$ for every $\nu = 1, 2, ..., j_m - 1$,

3° For every point x of K^n and every $m = 1, 2, ...$ there exists an index ν_x such that $\varrho(a_{m,\nu_x}, x) < \frac{1}{m}$.

Now let $a_0, a_1, ..., a_\nu, ...$ denote the sequence which we obtain, if we write first the points $a_{1,1}, a_{1,2}, ..., a_{1,j_1}$ (in the given order), then the points $a_{2,1}, a_{2,2}, ..., a_{2,j_2}$ and so on. Let

$$a_\nu = \{0, \xi_{\nu,2}, \xi_{\nu,3}, ...,\}.$$

Setting

$$b_\nu = \left\{\frac{1}{\nu}, \xi_{\nu,2}, \xi_{\nu,3}, ...\right\}$$

and denoting by L_ν the segment in H with the end points $b_{\nu-1}$ and b_ν, for $\nu = 1, 2, ...$, let us consider the sets

$$L = \bigcup_{\nu=1}^{\infty} L_\nu \quad \text{and} \quad Y = K^n \cup L.$$

It follows by our construction, that L is homeomorphic with the segment without one of its end points and that

$$K^n \cup L = 0 \text{ and } \bar{L} = Y.$$

Consequently, Y is a continuum and $\dim Y = \dim K^n = n$.

Since K^n does not contain any simple arc, we infer easily that every retract of Y is either a simple arc or a point. It follows that Y is an $\mathcal{R}$-neighbour of X on the right.

THEOREM 2. *There exist a 1-dimensional ANR-set* *) *X and a 2-dimensional polytope Y which are $\mathcal{R}$-neighbours.*

Proof. Let X be the universal dendrite, that is a dendrite **) containing topologically every other dendrite [3], and let Y denote the 2-dimensional disk. Evidently, $X \underset{\mathcal{R}}{<} Y$ and for every space Z such that $X \underset{\mathcal{R}}{\leqslant} Z \underset{\mathcal{R}}{\leqslant} Y$, Z is a plane AR-set of dimension 1 or 2. If $\dim Z = 1$, then Z is a dendrite and, consequently, Z is topologically included in X. Since Z is an AR-set, we infer that Z is homeomorphic to a retract of X, hence $Z \underset{\mathcal{R}}{\leqslant} X$ and finally $X \underset{\mathcal{R}}{=} Z$. If $\dim Z = 2$, then Z contains a disk and we infer that Y is homeomorphic to a retract of Z, hence $Y \underset{\mathcal{R}}{\leqslant} Z$ and we infer that $Y \underset{\mathcal{R}}{=} Z$.

*) By *ANR*-sets we understand here compact spaces homeomorphic with retracts of open subsets of the Hilbert space. Compact spaces homeomorphic to retracts of the whole Hilbert space are said to be *AR*-sets.

**) By dendrite we understand a locally connected continuum which does not contain any simple closed curve. Every dendrite is an *AR*-set.

THEOREM 3. *Let X and Y be two polytopes of positive dimensions. If X and Y are $\mathcal{R}$-neighbours, then dim X = dim Y.*

Proof. Let us assume that $X \underset{\mathcal{R}}{\leqslant} Y$ and let

$$m = \dim X; \quad n = \dim Y.$$

We have to show that if $m < n$, then there exists a polytope Z satisfying the condition

(1) $$X \underset{\mathcal{R}}{<} Z \underset{\mathcal{R}}{<} Y.$$

Evidently, we can assume that X is a retract of Y.

Consider a component Y_0 of Y satisfying the condition

$$\dim Y_0 = n,$$

and let us pick up a point a_0 lying in Y_0.

If $X \cap Y_0 = 0$, then it suffices to set

$$Z = X \cup (a_0)$$

in order to obtain a polytope Z satisfying (1).

If $X \cap Y_0 \neq 0$, then we consider a triangulation T of X. Let k denote the number of all m-dimensional simplexes in the triangulation T. Since $\dim Y_0 = n > m$, there exists in Y an n-dimensional simplex $\Delta \subset Y_0 - X$. Let us consider in Δ a polytope A being union of $k+2$ m-dimensional simplexes $\Delta_0, \Delta_1, \ldots, \Delta_{k+1}$ having an $(m-1)$-dimensional face in common. Since the triangulation T of X contains only k simplexes of dimension m, we infer easily that A is not homeomorphic to any subset of X.

Now let us consider a simple arc $L \subset Y$, its end points a and b belonging to A and X, respectively, and such that

$$L \cap (X \cup A) = (a) \cup (b).$$

Consider the set

$$Z = X \cup A \cup L.$$

Since X is a retract of Y, there exists a continuous map f of Y onto X, which is on X the identity. The partial map

$$g = f|Z$$

is a retraction of Z to X. Consequently, X is a retract of Z and we have

$$X \underset{\mathcal{R}}{\leqslant} Z.$$

But the set $A \subset Z$ is not homeomorphic to any subset of X. Hence,

(2) $$X \underset{\mathcal{R}}{<} Z$$

On the other hand, let us observe that the set $A \cup L$ is an AR-set. Consequently, there exists a continuous function $\varphi(x, t)$ defined for $(x, t) \in (A \cup L) \times \langle 0, 1 \rangle$, satisfying the conditions:

$\varphi(x,t) \in A \cup L$ for every $(x,t) \in (A \cup L) \times \langle 0,1 \rangle$,
$\varphi(x,0) = x$ for every $x \in A \cup L$,
$\varphi(x,1) = b$ for every $x \in A \cup L$,
$\varphi(b,t) = b$ for every $0 \leqslant t \leqslant 1$.

Setting

$g_t(x) = g\varphi(x,t)$ for every $(x,t) \in (A \cup L) \times \langle 0,1 \rangle$,
$g_t(x) = x$ for every $(x,t) \in X \times \langle 0,1 \rangle$,

we get a homotopy $\{g_t\}$ joining in the functional space Z^Z of all continuous maps of Z into Z, the retraction $g = g_0$ of Z to X with the retraction g_1 of Z to X satisfying the condition

$$g_1(A \cup L) = b.$$

On the other hand, if we set

$g'_t(x) = \varphi(x,t)$ for every $(x,t) \in (A \cup L) \times \langle 0,1 \rangle$,
$g'_t(x) = x$ for every $(x,t) \in X \times \langle 0,1 \rangle$,

we get a homotopy $\{g'_t\}$ joining in Z^Z the identity g'_0 with the retraction $g'_1 = g_1$. It follows that the retraction $g_0 = g = f|Z$ is homotopic in Z^Z to the identity. Since f is an extension of g_0 over Y, with values on Z, and since Z is a polytope, we infer that the identity on Z has also an extension over Y with values on Z, i.e. Z is a retract of Y. Hence,

$$Z \underset{\mathcal{R}}{\leqslant} Y.$$

But $\dim Y = n > m = \dim Z$. Consequently.

(3) $$Z \underset{\mathcal{R}}{<} Y.$$

It follows by (2) and (3) that X and Y are not $\mathcal{R}$-neighbours and our proof is concluded.

PROBLEM 1. *Is it true that the difference between the dimensions of two ANR-sets which are $\mathcal{R}$-neighbours is necessarily < 2?*

PROBLEM 2. *Let $X_0, X_1, \ldots, X_k$ be a finite sequence of spaces such that X_{i-1} and X_i are $\mathcal{R}$-neighbours for every $i = 1, 2, \ldots, k$ and that X_0 and X_k are polytopes of positive dimensions. Is then necessarily* $\dim X_0 = \dim X_k$?

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MATHEMATICS

On Models of Set Theory with Urelements

by

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Presented by A. MOSTOWSKI on May 23, 1960

The main result of this note is that the statement "For every function f the cardinality of the range of f is not greater than the cardinality of the domain of f" is not provable from the usual axioms of set theory with urelements without the axiom of choice, but with the ordering axiom added. Also some results concerning certain models of set theory with urelements will be mentioned. This note is mainly an answer to some problems raised, among others, by Fraïsse [1]. We shall use here the terminology and notations of Mostowski [2].

For cardinals $\mathfrak{m}, \mathfrak{n}$, $\mathfrak{m} * \geqslant \mathfrak{n}$ stands for "there exists a function f which maps a set of cardinality $\mathfrak{m}$ onto a set of cardinality $\mathfrak{n}$". For a non-zero finite n we denote with $K^{(n)}$ the set of all ordered n-tuples of distinct members of K.

LEMMA 1. *Let $\mathfrak{G}$ be a group of permutations of K such that for every finite n and every distinct $x, x_1, ..., x_n$ there exists a $\varphi \in \mathfrak{G}$ such that $\varphi(x_i) = x_i$ for $1 \leqslant i \leqslant n$ and $\varphi(x) \neq x$. Let A be a finite subset of K. Let $f \in \mathfrak{N}_{\mathfrak{G}}(A)$ be a function (i.e., $\langle uv \rangle \in f \wedge \langle uw \rangle \in f \to v = w$); then, if $\langle uv \rangle \in f$, $u \in K^{(p)}$ and $v \in K^{(q)}$, then $q \leqslant p + \overline{\overline{A}}$.*

Proof. Let $u = \langle u_1, ..., u_p \rangle$, $u_i \neq u_j$ for $i \neq j$, $v = \langle v_1, ..., v_q \rangle$, $v_i \neq v_j$ for $i \neq j$. Assume $q \geqslant p + \overline{\overline{A}}$; therefore there exists a k, $1 \leqslant k \leqslant q$, such that $v_k \notin A \cup \{u_1, ..., u_p\}$. Let φ be a member of $\mathfrak{G}$ such that φ is the identity on $A \cup \{u_1, ..., u_p\}$ and $\varphi(v_k) \neq v_k$. Obviously, $\varphi \in \mathfrak{G}(A)$ and, since $f \in \mathfrak{N}_{\mathfrak{G}}(A)$, we have $|\varphi, f| = f$. Since $\langle uv \rangle \in f$ we have $\langle |\varphi, u| \, |\varphi, v| \rangle = |\varphi, \langle uv \rangle| \in |\varphi, f| = f$. Since $\varphi(u_j) = u_j$, for $1 \leqslant j \leqslant p$, $|\varphi, u| = u$ and thus $\langle u \, |\varphi, v| \rangle \in f$. Since f is a function and $\langle uv \rangle \in f$, we have $v = |\varphi, v|$, contradicting $\varphi(v_k) \neq v_k$.

THEOREM 1. *Let $\mathfrak{G}$ be a group of permutations of K as required in Lemma 1 and let $M \in \mathfrak{N}(\mathfrak{G})$ contain only finite sets. Then in the model determined by $\mathfrak{B}_{\mathfrak{G}, M}$ we have*

$$\overline{\overline{\sum_{n \in \omega - \{0\}} K^{(n^2)}}} * \geqslant \overline{\overline{\sum_{n \in \omega - \{0\}} K^{(n)}}}, \quad \text{but} \quad \overline{\overline{\sum_{n \in \omega - \{0\}} K^{(n^2)}}} < \overline{\overline{\sum_{n \in \omega - \{0\}} K^{(n)}}}.$$

Proof. Let g be a function defined on $\sum_{n\in\omega-\{0\}} K^{(n^2)}$ by $g(\langle x_1,\ldots,x_{n^2}\rangle)=\langle x_1,\ldots,x_n\rangle$.

The range of g is $\sum_{n\in\omega-\{0\}} K^{(n)}$ and it is easily seen that $g\in\mathfrak{K}_{\mathfrak{G}}(\emptyset)$ *) and, hence, that $g\in\mathfrak{W}_{\mathfrak{G},M}$, i.e., $\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2)}}}*\geqslant\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n)}}}$ in the model.

Obviously, $\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2)}}}\leqslant\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n)}}}$ in the model. If $\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2)}}}=\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n)}}}$ in the model, then there is a one-one function $f\in\mathfrak{W}_{\mathfrak{G},M}$ mapping $\sum_{n\in\omega-\{0\}} K^{(n)}$ onto $\sum_{n\in\omega-\{0\}} K^{(n^2)}$. Since $f\in\mathfrak{W}_{\mathfrak{G},M}$, there exists an $A\in M$ such that $f\in\mathfrak{K}_{\mathfrak{G}}(A)$. By the assumption of the Theorem 1, A is finite; we put $A=m$.

Let $u\in K^{((m+1)^2+m+1)}$ and let $v=f(u)\in K^{(q^2)}$. By Lemma 1, $q^2\leqslant[(m+1)^2+m+1]+m<(m+2)^2$. Since, as can be easily seen, also $f^{-1}\in\mathfrak{K}_{\mathfrak{G}}(A)$, we have, again by Lemma 1, $(m+1)^2+m+1\leqslant q^2+m$, hence $q^2>(m+1)^2$, which is a contradiction.

Since the model constructed in [2] satisfies the requirements of Theorem 1, we have:

COROLLARY 1. *If von-Neumann's set theory is consistent, then in Mostowski's axiom system* $\mathfrak{S}$ *with the ordering axiom added one cannot prove the statement "for any two cardinals* $\mathfrak{m},\mathfrak{n}$, *if* $\mathfrak{m}*\geqslant\mathfrak{n}$, *then not* $\mathfrak{n}>\mathfrak{m}$".

In essentially the same way as for Theorem 1 and Corollary 1 we have:

THEOREM 2. *Let* $\mathfrak{G}$ *and* M *be as in Theorem 1. Then in the model determined by* $\mathfrak{W}_{\mathfrak{G},M}$ *we have*

$$\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2)}}}*\geqslant\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2+n)}}}\quad\text{and}\quad\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2+n)}}}*\geqslant\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2)}}},$$

but $\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2)}}}$ *and* $\overline{\overline{\sum_{n\in\omega-\{0\}} K^{(n^2+n)}}}$ *are incomparable.*

Thus' if von-Neumann's set theory is consistent, one cannot prove in the system $\mathfrak{S}$ *with the ordering axiom added the statement "for any two cardinals* $\mathfrak{m},\mathfrak{n}$, *if* $\mathfrak{m}*\geqslant\mathfrak{n}$ *and* $\mathfrak{n}*\geqslant\mathfrak{m}$, *then* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are comparable*".

Let $\mathfrak{S}^*$ be a theory which is like $\mathfrak{S}$, only that it has element variables only (i.e., it has no class variables) **). We shall regard models of $\mathfrak{S}^*$ which are like the general models of Mostowski [2], except for the following differences: first, K is now a set of any infinite cardinality [4]; second, the members of K are urelements. We take $\mathfrak{G}$ to be the group of all permutations of K and M — the set of all finite subsets of K.

Such a model will be denoted by $\mathfrak{W}_{(K)}$.

*) $\emptyset$ denotes here the void set.

**) Such a system was used, essentially, by Mostowski [3].

THEOREM 3. *Let K_1, K_2 be any infinite sets of urelements. Then $\mathfrak{W}_{(K_1)}$, $\mathfrak{W}_{(K_2)}$ are elementarily equivalent.*

THEOREM 4. *Let K_1, K_2 be infinite sets of urelements and let $K_1 \leqslant K_2$. Then $\mathfrak{W}_{(K_1)}$ is isomorphic to any elementary subsystem $\mathfrak{W}'$ of $\mathfrak{W}_{(K_2)}$ (i.e., a subsystem $\mathfrak{W}'$ such that $\mathfrak{W}_{(K_2)}$ is an elementary extension of $\mathfrak{W}'$).*

These theorems are proved by means of a ternary relation $S(f, x, y)$, where x is in $\mathfrak{W}_{(K_1)}$, y in $\mathfrak{W}_{(K_2)}$ and f is a one-one mapping of a finite subset A_1 of K_1 onto a finite subset A_2 of K_2. If f maps A_1 onto A_2, then $S(f, x, y)$ holds if and only if $x \in \mathfrak{K}_{\mathfrak{W}_1}(A_1)$, $y \in \mathfrak{K}_{\mathfrak{W}_2}(A_2)$ and x has the "same structure" with respect to K_1 and the members of A_1, as y has with respect to K_2 and the f-images of the members of A_1. The following lemmas are then established:

LEMMA 5. *If f maps A_1 onto A_2, then for every $x \in \mathfrak{K}_{\mathfrak{W}_1}(A_1)$ there exists exactly one $y \in \mathfrak{K}_{\mathfrak{W}_2}(A_2)$ such that $S(f, x, y)$ and for every $y \in \mathfrak{K}_{\mathfrak{W}_2}(A_2)$ there exists exactly one $x \in \mathfrak{K}_{\mathfrak{W}_1}(A_1)$ such that $S(f, x, y)$.*

LEMMA 6. *Let $\Theta(v_1, \ldots, v_n)$ be any formula of $\mathfrak{S}^*$ with no free variables except $v_1, \ldots, v_n$. If $x_1, \ldots, x_n$ are in $\mathfrak{W}_{(K_1)}$ and $y_1, \ldots, y_n$ in $\mathfrak{W}_{(K_2)}$ and f is such that $S(f, x_i, y_i)$, $1 \leqslant i \leqslant n$, then $\Theta(x_1, \ldots, x_n)$ holds in $\mathfrak{W}_{(K_1)}$ if and only if $\Theta(y_1, \ldots, y_n)$ holds in $\mathfrak{W}_{(K_2)}$.*

Theorem 3 follows from Lemmas 5 and 6 immediately. Theorem 4 is obtained as follows: Let g be a one-one mapping of K_1 into K_2. G is the function defined as follows: given x in $\mathfrak{W}_{(K_1)}$ let A be the minimal set such that $x \in \mathfrak{K}_{\mathfrak{W}_1}(A)$ and let $g \restriction A$ be the restriction of g to A. Then $G(x)$ is such an y for which $S(g \restriction A, x, y)$ holds. G is the function mapping $\mathfrak{W}_{(K_1)}$ isomorphically onto an elementary subsystem of $\mathfrak{W}_{(K_2)}$.

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On the Polarization of Neutrons from Muon Capture by the Nucleus

by

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Cini and Gatto [1] calculated the polarization of neutrons emitted from the nucleus as a result of capture of unpolarized μ-mesons; they found that if the weak interaction is $V-A$, then the fastest neutrons from the capture are almost totally polarized in opposite direction to their momenta. These authors also estimated (using an optical model) the depolarization of these neutrons caused by their rescattering before escaping from the nucleus: for 14 Mev neutrons from muon absorption in ^{12}C the depolarization is about 0.2.

In the present note the same problem is examined taking into account the virtual pion corrections, namely the effective *PS*-interaction and the interaction with derivatives of the lepton fields [2]; the latter is appreciable, if the coupling of a conserved vector current of strongly interacting particles to the lepton current is postulated [3].

Proceeding from the universal weak $V-A$ interaction we obtain the following effective nonrelativistic transition matrix element for muon capture by a "physical" proton moving with the momentum $\vec{p}$ [4]:

$$\mathcal{M} = \frac{G}{\sqrt{2}} f_V \frac{1}{\sqrt{2}} \left(1 - \vec{\sigma}_L \frac{\vec{k}}{k}\right)(a + \vec{b}\vec{\sigma}_N),$$

where

$$a = 1 + \frac{\vec{p} + \vec{p}_n}{2M} \vec{\sigma}_L,$$

$$\vec{b} = -\left[\lambda\left(\vec{\sigma}_L + \frac{\vec{p} + \vec{p}_n}{2M}\right) + \xi \frac{\vec{p} - \vec{p}_n}{2M} i(\varkappa + 1) \frac{\vec{p} - \vec{p}_n}{2M} \times \vec{\sigma}_L\right].$$

Here the terms of order v_{nucl}/c are retained. Notations: $\vec{\sigma}_N$, $\vec{\sigma}_L$ — the the matrix elements of Pauli matrices taken for nucleons or leptons, respectively; $\vec{p}_n$, $\vec{k}$—momenta of neutron and neutrino, respectively; $\frac{G}{\sqrt{2}}$ is

the universal coupling constant, $f_V = C_V^{\text{eff}} \Big/ \frac{G}{\sqrt{2}}$, $\lambda = -C_A^{\text{eff}}/C_V^{\text{eff}}$. $\xi = -C_P^{\text{eff}}/C_V^{\text{eff}}$, $\varkappa/2M = C_{der}^{\text{eff}}/C_V^{\text{eff}}$, where C_V^{eff}, C_A^{eff}, C_P^{eff} and C_{der}^{eff} are the effective constants of the V, A, PS and derivative couplings, respectively; M is the nucleon mass, v_{nucl} is the nucleon velocity.

The neutron polarization $\vec{P}_n$ given by

$$Sp(\mathcal{M}^+\mathcal{M})\,\vec{P}_n = Sp(\mathcal{M}^+\,\vec{\sigma}_N\mathcal{M}) \tag{1}$$

must be averaged over the motion of protons in the nucleus, for which we assume the Fermi gas model: $0 \leqslant p \leqslant p_F = \sqrt{2ME_F}$, $E_F = 33\text{MeV}$. For this purpose we multiply (1) by

$$d^3p\,d^3p_n\,dE_f = p^2\,dp\,d\varphi\,\frac{d(-\cos\vartheta)}{dE_f}\,p_n^2\,dp_n\,d\Omega_n = p^2\,dp\,d\varphi\,\frac{k}{pp_n}\,Mp_n\,dE_n\,d\Omega_n =$$

$$= M^2 k\,dk\,d\varphi\,dE_n\,d\Omega_n.$$

In the above transitions we used the conservation of the momentum and energy: $\frac{p^2}{2M} + m = k + E_n$, $\vec{p} = \vec{k} + \vec{p}_n$, where E_n is the neutron energy inside the nucleus, m is the muon mass. $\vartheta = \sphericalangle(\vec{p}, \vec{p}_n)$: the muon is assumed to be captured at rest.

The emitted neutrons obey the Pauli exclusion principle. Then must be $p_n > p_F$, and for this reason the lower integration limit for p is some $p_{\min} > 0$. We introduce a more convenient integration variable k, and it may be shown that $k_1 < k < k_2$, $k_1 = M + \sqrt{2ME_n} - [(M + \sqrt{2ME_n})^2 - 2Mm]^{1/2}$, $k_2 = E_F + m - E_n$.

Therefore, the neutron polarization averaged over the momentum spread of the nuclear protons is given by

$$\vec{P}_n = \int Sp(\mathcal{M}^+\vec{\sigma}_N\mathcal{M})\,k\,dk\,d\varphi \Big/ \int Sp(\mathcal{M}^+\mathcal{M})\,k\,dk\,d\varphi.$$

Using the general form of $Sp(\mathcal{M}^+\vec{\sigma}_N\mathcal{M})$ and of $Sp(\mathcal{M}^+\mathcal{M})$, we integrate over φ:

$$\int_0^{2\pi} Sp(\mathcal{M}^+\vec{\sigma}_N\mathcal{M})\,d\varphi = \int_0^{2\pi} |a(k,E_n)\,\vec{p} + b(k,E_n)\,\vec{p}_n|\,d\varphi = 2\pi(ap\cos\vartheta + bp_n)\,\vec{n},$$

$$\int_0^{2\pi} Sp(\mathcal{M}^+\mathcal{M}) = \int_0^{2\pi} c(k,E_n)\,d\varphi = 2\pi c, \quad (\vec{n} = \vec{p}_n/p_n).$$

Then

$$\vec{P}_n = \int_{k_1}^{k_2} (ap\cos\vartheta + bp_n)\,k\,dk \int_{k_1}^{k_2} ck\,dk = |R(E_n)/I(E_n)|\,\vec{n} = P_n(E_n)\,\vec{n}.$$

For the functions $R(E_n)$, $I(E_n)$ we obtain

$$-\sqrt{\frac{2M}{E_n}}R(E_n) = m\left\{4\lambda+\frac{1}{E_n}\left[-(\mu+\xi+\lambda)m+2(\lambda^2+\lambda)M\right]\right\}(k_2-k_1)+$$

$$+\left\{4\lambda+\frac{1}{E_n}\left[(2\lambda\mu+2\mu+\xi)m-2(\lambda^2+\lambda)M\right]\right\}\frac{k_2^2-k_1^2}{2}+$$

$$+\frac{1}{2M}\left\{4\mu+\frac{1}{E_n}\left[(\mu^2-2\mu-\xi-\lambda)m-2(2\lambda\mu+\mu-\lambda^2-2\lambda)M\right]\right\}\frac{k_2^3-k_1^3}{3}+$$

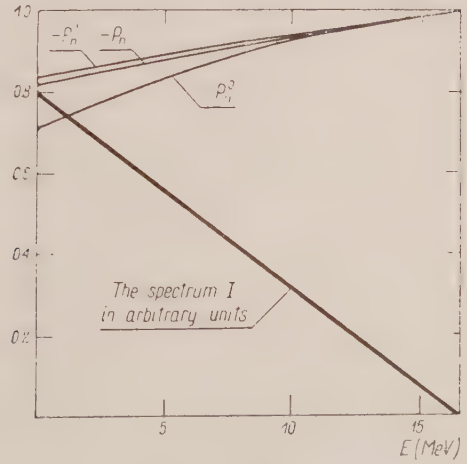
$$+\frac{1}{2M}(2\lambda\mu+2\mu-\lambda-\mu^2)\frac{1}{E_n}\frac{k_2^4-k_1^4}{4}+\frac{1}{(2M)^2}(\mu^2-\mu)\frac{1}{E_n}\frac{k_2^5-k_1^5}{5};$$

$$I(E_n) = 2(1+\lambda^2)m(k_2-k_1)+\left[\lambda^2-1-(\lambda\xi+\lambda^2+1)\frac{m}{M}+2(1+\lambda^2)\frac{E_n}{M}\right]\frac{k_2^2-k_1^2}{2}+$$

$$+\frac{1}{M}(1+\lambda^2+2\lambda\mu)\frac{k_2^3-k_1^3}{3}+\frac{1}{(2M)^2}(\xi^2+2\mu^2-\lambda^2-1)\frac{k_2^4-k_1^4}{4};$$

$$\mu = 1+\varkappa.$$

The obtained neutron spectrum I and longitudinal polarization P_n are represented in the Fig. as functions of the energy of emitted neutrons outside the nucleus: $E = E_n - E_F - E_B$. E_B is the binding energy for the last neutron. Here we assumed the values of the effective coupling constants as $\lambda = 1.29$, $\xi = 10.3$, $\varkappa = 3.7$. These values follow from the results of



Goldberger and Treiman's paper [2] on the comparison of the β-decay and μ-capture coupling constants, if we adopt the conserved vector current hypothesis and use the experimental results on β-decay and on tle electromagnetic formfactors of nucleons. As it is seen, P_n varies with the neutron energy from -0.82 to -0.99, and its average over the

spectrum is -0.91. The curve P'_n is the neutron polarization without the conserved vector current assumption; the difference between P_n and P'_n is quite small. P_n^0 is the neutron polarization for the pure $V-A$ interaction taken from [1].

Comparing this result with that of Cini and Gatto we conclude that, in the framework of the universal Fermi interaction theory, measurement of the longitudinal polarization of neutrons from muon capture in nuclei cannot apparently clarify the question of the virtual pion effects in this process.

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Notes on the Unified Field Theory

by

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Introduction

The close connexion of geometry and physics and the deduction of the equations of motion from the field equations without any additional principle are two great achievements of the general relativity theory. The electromagnetic field as one of the sources of the gravitational field enters, however, into the field equations of this theory through the energy tensor which has been always considered by Einstein as a provisional means of representing matter.

The unification of the gravitational and of the electromagnetic fields into one geometrical field became possible, after the asymmetric affine geometry had been substituted for the Riemannian geometry of the general relativity theory. The hope that the asymmetric unified field theory will explain also the structure of the elementary particles seems to have been disappointing.

These two factors are the starting point of the unified field theory of gravitation and electricity built up by the present author [1] and now supplemented by the following three notes.

1. The postulates of the unified field theory of gravitation and electricity and the choice of the Hamiltonian

It has been shown [2] that the homogeneous Hamiltonian proposed by Einstein [3a] must be replaced by a heterogeneous one, if in a strong gravitational and a weak electromagnetic field the field equations deduced therefrom are to take the form corresponding to the field equations of the general relativity theory of Einstein-Maxwell.

In order to restore the homogeneity of the Hamiltonian under the simultaneous fulfilment of the just mentioned postulate, the author made an attempt to replace the well-known λ-transformation of Einstein

$$\Gamma^{\varepsilon *}_{\mu\nu} = \Gamma^{\varepsilon}_{\mu\nu} + \delta^{\varepsilon}_{\mu}\lambda_{\nu}$$

by a more general one

$$\Gamma^{\varepsilon *}_{\mu\nu} = \Gamma^{\varepsilon}_{\mu\nu} + 2\,\delta^{\varepsilon}_{\mu} L_{\nu} + M^{\varepsilon}_{\mu\nu},$$

$M^{\varepsilon}_{\mu\nu}$ being a tensor containing the skew-symmetric part of fundamental tensor. This attempt, however, inspired by the papers of Eisenhart [4], proved unsuccessful. It seems therefore to be more natural to start from the very beginning from a heterogeneous Hamiltonian without introducing the λ-transformation.

The Hamiltonian of the unified field theory of gravitation and electricity may be then determined by the following four postulates:

(1) The corner stone of the four-dimensional asymmetric field theory is the asymmetric covariant fundamental tensor $g_{\mu\nu}$ from which the asymmetric contravariant components of the tensor $g^{\mu\nu}$ and of the tensor density $\mathfrak{g}^{\mu\nu}$ may be deduced by means of the relations

$$g_{\mu\alpha}\, g^{\nu\alpha} = g_{\alpha\mu}\, g^{\alpha\nu} = \delta^{\nu}_{\mu}, \tag{1.1}$$

$$\mathfrak{g}^{\mu\nu} = g^{\mu\nu} \sqrt{(-\det g_{\alpha\beta})}, \tag{1.2}$$

δ^{ν}_{μ} being the Kronecker tensor.

(2) In a strong symmetric and a weak skew-symmetric field of the fundamental tensor the field equations of the unified field theory may differ from those of the general relativity theory of Einstein-Maxwell merely by negligeably small terms.

(3) The invariant Hamiltonian $\mathfrak{h}$ is formed by the asymmetric contravariant tensor density $\mathfrak{g}^{\mu\nu}$ and by the asymmetric covariant tensor $R^{*}_{\mu\nu}$:

$$\mathfrak{h} = \mathfrak{g}^{\mu\nu} R^{*}_{\mu\nu}. \tag{1.3}$$

The tensor $R^{*}_{\mu\nu}$ beside the fundamental tensor $g_{\mu\nu}$, depends explicitely also on two further fields, viz. on the asymmetric affine connexion $\Gamma^{\varepsilon}_{\mu\nu}$ and on the vector field L_{μ}.

(4) All the field equations have the property of transposition invariance, i.e. they still hold, even if we replace

$$g_{\mu\nu} \quad \text{by} \quad \tilde{g}_{\mu\nu} (\equiv g_{\nu\mu}) \quad \text{and} \quad \Gamma^{\varepsilon}_{\mu\nu} \quad \text{by} \quad \tilde{\Gamma}^{\varepsilon}_{\mu\nu} (\equiv \Gamma^{\varepsilon}_{\nu\mu}).$$

The first postulate is the fundamental postulate formulated by Einstein determining the characteristic features of the theory. The symmetric and the skew-symmetric parts of the fundamental tensors $g_{\underline{\mu\nu}}$, $g_{\underset{\smile}{\mu\nu}}$, respectively, which transform independently from each other, are unified by Eq. (1.1).

The second postulate is the heuristic postulate of the highest significance for the deduction and physical interpretation of the field equations. It represents the correspondence principle of the unified field theory in a stronger formulation than that of Korsunoğlu [5].

The third postulate deals with the form of the Hamiltonian.

There is no doubt that it must not necessarily have an analytically meaningless form (e.g. the square of the length must not be added to the trace of another tensor, as it is done in the general relativity theory of Einstein-Maxwell [6]). We require therefore that it be given by the inner product of a contravariant tensor density and a covariant tensor. The most simple and most natural contravariant tensor density which can be formed by the fields $g_{\mu\nu}$, $\Gamma^{\varepsilon}_{\mu\nu}$, L_μ is that determined by Eq. (1.2). We have postulated further that the covariant tensor $R^*_{\mu\nu}$ should contain, besides the asymmetric fundamental tensor $g_{\mu\nu}$, also the asymmetric affine connexion $\Gamma^{\varepsilon}_{\mu\nu}$, because it replaces here the connexion $\left\{{\varepsilon \atop \mu\nu}\right\}$ of the general relativity theory, and the covariant vector L_μ, which takes here the role of the vector potential A_μ of the Maxwell theory. By this inductive way these two fields have been thus incorporated into the unified field theory.

On the basis of postulate (2) we assume that the affine connexion $\Gamma^{\varepsilon}_{\mu\nu}$ appears in the unified field theory through the contracted Riemann tensor $R_{\mu\nu}$:

$$R_{\mu\nu} = -\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}. \tag{1.4}$$

As usual, the comma denotes the partial derivative.

A covariant skew-symmetric tensor of the second rank can be formed from the covariant vector L_μ only by the rotation:

$$(L_{\nu,\mu} - L_{\mu,\nu}). \tag{1.5}$$

The condition of the skew-symmetry assures that this tensor does not appear in the pure gravitational field.

The last tensor of the second rank which may occur in the tensor $R^*_{\mu\nu}$ is given by the expression

$$(ag_{\mu\nu} + bg_{\nu\mu}), \tag{1.6a}$$

or, in other notation, by

$$(a\underline{g}_{\mu\nu} + \beta\underset{\smile}{g}_{\mu\nu}). \tag{1.6b}$$

The asymmetric tensor $R^*_{\mu\nu}$ is now formed by adding the preceding three tensors:

$$R^*_{\mu\nu} = R_{\mu\nu} - 2(L_{\nu,\mu} - L_{\mu,\nu}) + (a\underline{g}_{\mu\nu} + \beta\underset{\smile}{g}_{\mu\nu}). \tag{1.7}$$

It is the most simple covariant tensor of the second rank which can be written by the quantities: $\Gamma^{\varepsilon}_{\mu\nu}$, L_μ, and $g_{\mu\nu}$. At the same time it represents the most natural generalization of the corresponding tensor of the general relativity theory.

The factor -2 at the second term on the right-hand side of Eq. (1.7) is an unsignificant constant which has been introduced in order to obtain the deduced field equations in a simple form.

The coefficients α, β have been introduced because of dimensional considerations. They have the dimension: $(\text{length})^{-2}$. We assume that they are constant, lest the theory should be complicated without well-grounded reasons. The first of them, α, can equal zero, but the second, β, may never vanish.

The fourth postulate, proposed by Einstein, replaces the symmetry in the field equations. From the physical standpoint it expresses the indifference of the theory to the sign of electricity. It is satisfied by the Hamiltonian given by Eqs. (1.3) and (1.7).

2. On the number of universal natural constants in the field theory of gravitation and electricity

As shown in [1], this unified field theory contains three universal natural constants, the velocity of light, the Newton's gravitation constant, and the new constant d related to α and β by the formula

$$d = (\beta - \alpha)^{-\frac{1}{2}} \neq 0 \tag{2.1}$$

and determining the degree in which the symmetric and the skew-symmetric parts of the fundamental tensor influence each other. To these three constants the cosmological constant

$$\lambda = -2\alpha \tag{2.2}$$

has been added as the fourth. However, this constant λ must be excluded from the theory because of the reasons given by Einstein [3b] and recently by the present author [7].

The unified field theory should be a genuine geometrical theory containing no physical constants. We satisfy this demand by putting

$$\alpha = 0, \quad \beta = d^{-2} = 2a = -2b = 1. \tag{2.3}$$

All the quantities of the theory are then measured by dimensionless numbers.

As was to be expected, the unified field theory excludes also the last arbitrariness in the choice of the unit of length which had as yet remained in the general relativity theory. The unit of length is now uniquely fixed by the theory.

The natural constant d, the velocity of light, the Newton's gravitation constant, and the dielectric constant of vacuum will be successively introduced into the theory, as we wish to measure the length, the time, and the integration constants, physically interpreted, by freely chosen units of length, of time, of mass, and of electricity, respectively.

3. The meaning and the limits of the unified field theory of gravitation and electricity

Instead of the inductive way by which the Hamiltonian has been determined in the 1st note we may start from the variational principle

$$(1.3) \qquad 8\pi\delta W_1 = \delta \int \mathfrak{g}^{\mu\nu}\left[R_{\mu\nu} + 2\,(L_{\mu,\nu} - L_{\nu,\mu}) + \frac{1}{2}\,(g_{\mu\nu} - g_{\nu\mu})\right] d\tau = 0\,,$$

supplemented by the definition relations (1.1), (1.2), (1.4), and by

$$d\tau = dx_1\,dx_2\,dx_3\,dx_4$$

and consider it as a fundamental natural law. The whole classical theory of gravitation and electricity then follows therefrom by a purely deductive method.

The main aim of this unified field theory is to describe the motion of the singular points in the field. The mathematical deduction of the equations of motion will be greatly facilitated, if we replace the variational principle (3.1) by the following one

$$(3.2) \qquad \delta W_1 + \delta W_2 = 0\,,$$

where W_2 denotes the action due to all the singular points representing the particles [1].

From the mathematical point of view this unified field theory is far more complicated than the general relativity theory. At the same time, the predicted differences between them take measurable values only in the domain of elementary particles. In the region outside them we shall always use with very good accuracy the Maxwell approximation of the unified field theory [1], [2] which coincides with the general relativity theory of Einstein-Maxwell. Both theories become identical, if the new natural constant d, which the hypothesis formulated by the present author in [1] associates with the elementary electric charge, vanishes.

The unified field theory under consideration forms a transition between the general relativity theory of the pure gravitational field, which may be considered as the only genuine field theory hitherto, and the future general unified field theory dealing, moreover, with the nuclear forces and the quantum phenomena. Its proper meaning lies in that it removes the grave objection to the general relativity theory of Einstein-Maxwell consisting in that the gravitational field and the electromagnetic field enter into it as logically distinct features [3c], [6].

Since we consider as the main task of this unified field theory and of the general relativity theory deduced therefrom the description of motion of the singular points, the energy concept as well as the energy

tensor *) are wholly superfluous. The energy concept becomes, however, a very useful tool for describing the physical reality within the field of a central body or the inertial co-ordinate systems of the special relativity theory.

Similarly as in the general relativity theory, also in the present unified field theory the particles are represented by singular points in the field which is an adequate means at this stage of development of the theory for assuring the elementarity of the particles in a non-linear theory, i.e. the independence of the properties of one particle from the others. The field equations show that the singular points may be physically interpreted as simple poles or multipoles with different mechanic, electric, and magnetic properties, but they give no account for the existence of a low number of elementary particles with definite physical properties.

The author is firmly convinced that here lie the limits of the unified field theory based on the asymmetric affine geometry. It seems that the problem of elementary particles and of the quantum phenomena with all their divergent difficulties can only be solved by a further generalization of the geometry consisting in an abandonement of the space-time continuum [3d], [8]. To achieve the solution in the framework of asymmetric affine geometry will be hardly possible.

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*) At this occasion we point out that the recent investigation of the author [7] proved that the introduction of the energy tensor of continuous matter into the field equations of the general relativity theory demands a careful examination in each particular case.

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On the Possibility of Measuring the Branching Ratio $R = \frac{(\pi^- + p \to \pi^0 + \pi^- + p)}{(\pi^- + p \to \pi^+ + \pi^- + n)}$ at Very High Energies

by

A. KRZYWICKI

Presented by L. INFELD on July 14, 1960

1. The aim of this paper is to examine the possibility of measuring the branching ratio $R = \frac{(\pi^- + p \to \pi^0 + \pi^- + p)}{(\pi^- + p \to \pi^+ + \pi^- + n)}$, at energies of the incident pion of the order of several Gev (such that the probability of production of two additional pions is important, viz $E > 4$ Gev). In measuring R we meet with a striking obstacle. In fact, neutral pions are not directly observable, so that actually we measure the quantity $R^{(\mathrm{exp})} = \frac{(\pi^- + p \to \pi^- + p + ?)}{(\pi^- + p \to \pi^- + \pi^+ + ?)}$ rather than R, as long as we do not look for secondary effects of π^0 s, which — for obvious reasons — is for high energies very difficult. However, for sufficiently high energies, $R^{(\mathrm{exp})}$ can appreciably differ from R. We shall try to express the $R^{(\mathrm{exp})}$ R ratio corresponding to certain selected events in terms of known observable quantities.

All essential assumptions adopted in our paper are put in sec. **2** (printed in italics). In sec. **3** an experiment is proposed. sec. **4** deals with calculations.

2. The experimentally observed forward collimation of final pions in pion, nucleon collisions [1]. [2] (two-prong stars) seems to support the idea of strong pion-pion interaction.

Consequently, in the following we assume strong pion-pion interaction.

The experiments proposed, however, will not be a direct verification of the existence of strong pion-pion interaction. We look for consequences of this assumption to see, which properties of this interaction are given by the experiment. From the point of view of the pion-pion interaction theory the quantity R is of interest, since it can give us valuable information concerning the isotopic properties of this interaction.

We limit ourselves to peripheral pion-nucleon collisions.

There are two principal reasons for this. First, we can expect that the role of pion-pion interaction increases with the peripherality of collision. Secondly, in this case one can more easily propose some soluble models corresponding to some definite mechanisms of interaction, which is much more difficult in the case of central collisions. We know that the concept of peripheral collision, although intuitively clear from the experimental point of view, is not a very precise one. Intuitively we imagine a collision to be peripheral, when (classically speaking) the impact parameter is large or — using more rigorous language of quantum mechanics — when scattering occurs in a state with large relative angular momentum of colliding particles. In considering peripheral collisions of nucleons I. Tamm, I. M. Dremin and D. S. Černiavski [3] used the physically plausible picture with a small momentum transfer (transferred momentum being carried by a single pion) between incident particles.

We also identify peripheral collisions with such collisions, where only a small amount of momentum of one incident particle is transferred to the other by a single intermediate particle.

Speaking in terms of graphs we shall call "peripheral" e.g. events symbolized by graphs having two "centres", each characterised by one external entering line and several external leaving lines; external lines leaving the centre take almost the entire momentum of the external entering line. Momentum is transferred from one centre to the other by a single intermediate meson. This identification may be justified but was not proved. One should note, however, that for an experiment we propose to select events according to the notion of momentum transfer, and not an impact parameter. Thus, our calculations are applicable to the proposed experiment, even if the collisions which we call peripheral are not peripheral in the strict sense.

We know that there is no sharp distinction between peripheral and central collisions. Neither are we sure that some typical events which we ascribe to peripheral collisions cannot result from a central one. Thus, every experiment in which events are selected according to the notion of peripherality, gives biased results. On the other hand, introduction of this idea seems to be the simplest way of explaining certain typical properties in a class of observable events. It is possible that some other mechanisms may produce these events, but we shall assume that peripheral collisions are predominantly responsible for them, i.e., *we assume that selecting experimentally observed stars, according to properties which are typical for peripheral collisions, we really get a sample of stars in which the contamination of stars due to non-peripheral collisions is small.*

Concerning the interaction mechanism *we assume that the incident pion interacts with one and only one pion of the virtual cloud surrounding the target nucleon.*

As already pointed out [4] in this case, only an odd number of additional pions can be produced at the periphery of the nucleon. This follows from the fact that the matrix element between states (also virtual) containing even and odd numbers of pions (and only pions) always vanishes. In calculations we shall make two simplifying assumptions, the justification of which will be given further.

In one step of calculations we treat the virtual pion as if it were a real one.

We assume that the peripheral pion-nucleon collision is characterized by scattering of the incident and virtual pions in a low relative angular momentum state and that the pion-pion scattering amplitude either depends weakly on energy or is dominated by a resonance.

We assume that the interaction of the incident pion with the virtual one can lead to the excitation of the whole nucleon, an isobar of mass M (of course diffuse) being formed. This isobar in decaying produces additional pions. Moreover, *we assume that only pion-pion interaction and decay of the produced isobar are responsible for the production of additional pions in peripheral pion-nucleon collisions.*

Since the angular distribution of pions corresponding to many-prong stars is much more isotropic than that corresponding to two-prong stars [1], [2], *we assume that at the energy in question one can neglect the probability of production of three additional pions in a peripheral pion-nucleon collision.* (This condition fixes, of course, the upper limit of the energy for which our calculations are applicable). In other words, we shall calculate the ratio $R^{(\mathrm{exp})}/R$ where $R^{(\mathrm{exp})} = \frac{(\pi^- + p \to \pi^0 + \pi^- + p \text{ and } 2\pi^0 + \pi^- + p)}{(\pi^- + p \to \pi^+ + \pi^- + n \text{ and } \pi^0 + \pi^+ + \pi^- + n)}$ the mechanisms of one and two additional pions production being those symbolized by the graphs in Figs. 1a and 1b, respectively.

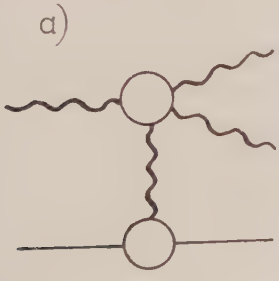

Fig. 1a

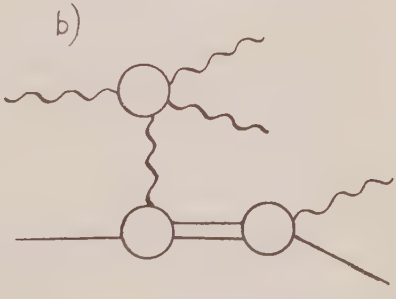

Fig. 1b

3. In order to select events, among which the contamination of non-peripheral collisions would be as small as possible, we impose the following restrictions on the momenta of charged particles:

We consider only stars, where (after the transformation to cms) final charged pions are emitted forward with momentum exceeding some value fixed beforehand, being in general a function of the angle of emission, that is $|\vec{k}_1| > h_1|\vec{k}_1|$, $\vec{k}_2 > h_2|\vec{k}_2|$, where h_1 and h_2 are some simple functions chosen by the experimenter; if a proton is present in the final state, we take into account only events in which the proton is emitted backwards (in cms), with momentum not smaller than that taken by a nucleon emitted forward in the reference frame of the decaying isobar which flies with the momentum (in cms) differing from the momentum of the incident proton by a quantity of the order of meson mass. We shall write this condition as $|\vec{t}| > h_3(\vec{t})$. This is not an entirely rigorous condition from the experimental point of view, the isobar mass being diffused. We must remember, however, that all the boundaries occurring in our paper are diffused as well as the energy of the incident beam.

The experiment which we want to propose consists in counting the relative frequency of occurrence of two kinds of stars (in lab. system), first those with two energetic prongs and, secondly, those with one energetic and one relatively slow prong, identifiable as a proton, which after transformation to cms *) fulfil the above mentioned conditions (these conditions in their numerical form must, of course, be fixed separately for each experiment).

We shall not specify the form of functions h_1 and h_2 for the sake of generality. Our opinion, however, is that the simplest experiment would consist in looking for events, where charged pions are emitted forward (in cms) with momentum larger than some value r independent of the angle of emission (the single condition imposed on r is that it should be larger than the maximal momentum of a pion flying forward (in cms) and emitted by the decaying isobar flying backwards (in cms).

4. The following notations will be used:

$\vec{p}$, p_0 — momentum and energy of the incident proton
$\vec{q}$, ω_0 — „ „ „ „ „ „ pion
$\vec{k}$, ω — „ „ „ „ „ virtual pion
$\vec{P}$, P_0 — „ „ „ „ „ isobar
$\vec{t}$, t_0 — „ „ „ „ „ final nucleon
$\vec{k}_1$, ω_1; $\vec{k}_2$, ω_2; $\vec{k}_3$, ω_3 — momenta and energies of final pions
M, m, μ — isobar, nucleon and pion masses
Θ — angle of emission of the final nucleon (Fig. 1a) or isobar (Fig. 1b).

*) Note that the probability of emission of a proton forward (in cms) and with considerable momentum is very small and can be neglected.

If we calculate the pion-nucleon vertex by perturbation theory, the cross-section for a process symbolized by the graph in Fig. 1a, taking our restrictions into account, is: *)

$$(1)\qquad \sigma_a = \frac{-(2\pi)^4 a g^2}{8 p_0 \omega_0} \int\limits_{|\vec{k}_1|>h_1} \frac{d_3 \vec{k}_1}{(2\pi)^2} \int\limits_{|\vec{k}_2|>h_2} \frac{d_3 \vec{k}_2}{(2\pi)^3} \int\limits_{|\vec{t}|>h_3} \frac{d_3 \vec{t}}{(2\pi)^2} \cdot \frac{A(q,-k)|^2}{\omega_1 \omega_2 t_0}$$

$$\cdot \frac{(m^2+(pt))}{|(t-p)^2+\mu^2|^2} \cdot \delta_4 (p+q-k_1-k_2-t) =$$

$$= \frac{-a g^2}{8\cdot(2\pi)^5 p_0 \omega_0} \cdot |A(q,-\vec{k})|^2 \cdot \int\limits_{|\vec{t}|>h} \frac{d_3 \vec{t}}{t_0} \cdot \frac{(m^2+(pt))}{|(t-p)^2+\mu^2|^2} \cdot F(t),$$

where

$$(2)\qquad F(t) = \int\limits_{|\vec{k}_1|>h_1} \frac{d_3 \vec{k}_1}{\omega_1} \int\limits_{|\vec{k}_2|>h_2} \frac{d_3 \vec{k}_2}{\omega_2} \cdot \delta_4 (p+q-k_1-k_2-t)$$

is a purely kinematic factor (see Appendix). $A(p, -k)$ describes the pion-pion vertex and a equals 1 or $^1/_2$ depending on whether we have in the final state a neutron or proton (of course, in these two cases we have different F's, since h_i corresponding to these cases are different).

To calculate the cross-section for the process symbolized by the graph in Fig. 1b we shall use the Dremin-Černiavski method. We have (when in the final state neutron is present)

$$(3)\qquad \sigma_b = \frac{(2\pi)^4 m^2}{16 p_0 \omega_0} \int\limits_{|\vec{k}_1|>h_1} \frac{d_3 \vec{k}_1}{(2\pi)^3} \int\limits_{|\vec{k}_2|>h_2} \frac{d_3 \vec{k}_2}{(2\pi)^3} \int \frac{d_3 \vec{k}_3}{(2\pi)^3} \int \frac{d_3 \vec{t}}{(2\pi)^3} \cdot \frac{1}{\omega_1 \omega_2 \omega_3 t_0}$$

$$\cdot \left| \frac{T(p,k) A(q,-k)}{(k^2+\mu^2)} + \frac{T(p,k') A(q,-k')}{(k'^2+\mu^2)} \right|^2 \cdot \delta_4 (p+q-k_1-k_2-k_3-t).$$

Here k' denotes the 4-momentum of the virtual pion in the case when a charged meson is emitted by the decaying isobar. When finally we have a proton, we can write an analogous expression, the matrix element being symmetrized with respect to the two final π^0's and a restriction imposed on the integration with respect to t. $T(p, k)$ describes the scattering of a pion (with momentum k) on a nucleon (with momentum p). Like Dremin and Černiavski, we treat the virtual meson as if it were a real one. It means that we connect T with the pion nucleon cross-section by the following equation

$$(4)\qquad J_{\pi n}\, \sigma_{\pi n} = \frac{(2\pi)^4 m^2}{4 p_0 \omega} \int \frac{d_3 \vec{k}_3}{(2\pi)^3} \int \frac{d_3 \vec{t}}{(2\pi)^3} \cdot \frac{|T(p,k)|^2}{\omega_3 t_0} \cdot \delta_4 (p+k-k_3-t).$$

*) In the case when k_i or t denotes momentum of a neutral particle $h_i \equiv 0$.

We can always choose h_1 and h_2 in such a manner as to be able to omit the term involving k', since our matrix element is such that the isobar flies almost always backwards, and therefore the maximal momentum (in cms) carried by a meson flying forward and produced by the decaying isobar is relatively small. We also neglect the interference term originating from the symmetrization with respect to π^0's in the case, when in the final state, a proton is present.

Thus, we get *)

$$(5)\qquad \sigma_b = \frac{1}{4\,p_0\,\omega_0}\int d_4 P \int\limits_{|\vec{k}_1|>h_1} \frac{d_3\vec{k}_1}{(2\pi)^2} \int\limits_{|\vec{k}_2|>h_2} \frac{d_3\vec{k}_2}{(2\pi)^3} \frac{|A(q,-k)|^2}{\omega_1\,\omega_2} \times$$

$$\times \frac{(p_0\,\omega\, J_{\pi n}\,\sigma_{\pi n})}{[(P-p)^2+\mu^2]^2}\cdot \delta_4(p+q-k_1-k_2-P) =$$

$$= \frac{|A(g,-\tilde{\tilde{k}})|^2}{4\,(2\pi)^6\,p_0\,\omega_0}\int d_4 P\, \frac{(p_0\,\omega\, J_{\pi n}\,\sigma_{\pi n})}{[(P-p)+\mu^2]^2} F(P)$$

(note that we take into account only protons emitted forward in the isobar rest system).

Let us assume first that the pion-pion scattering amplitude at the energy corresponding to our calculations sufficiently weakly depends on energy and angle, i.e.

$$\frac{A(q,-\tilde{k})|^2}{|A(q,-\tilde{\tilde{k}})|^2} \sim 1\,.$$

In fact, for the pion-nucleon collision to be really peripheral, the incident pion must interact with the limited region of the cloud. It means that the interaction must take place, when the relative angular momentum of the incident and cloud mesons is small. This does not necessarily mean that the low angular momentum interaction predominantes in the free-particle case. It can be that, if the incident pion interacts with the cloud pion "lying far" from its "path" through the cloud, the contamination of the interaction with nucleon core is so large that we cannot speak of peripheral collision. It is, of course, only a plausibility argument and not a proof. Using (1) and (5) we get

$$(6)\qquad \frac{R^{(\mathrm{exp})}}{R} = \frac{1+c_1}{1+c_2},$$

*) We do not write in this formula any restriction imposed on the domain of integration with respect to P. In fact, the term under the integral originating from the one-meson propagator automatically gives small probability to events characterised by large Θ and small $|\vec{P}|$. On the other hand it would be more consistent to integrate over $\Theta < \Theta_{\max}$ and $|\vec{P}| > |\vec{P}|_{\min}$. In the following we shall not write this condition explicitly in order to simplify the expressions. In practical calculations the use of such a condition depends on the accuracy required.

where

$$(7)\quad \begin{cases} c_1 = \dfrac{-1}{2\pi^2\left(\dfrac{g^2}{4\pi}\right)} \dfrac{\int d_4 P \dfrac{[p_0\,\omega\, J_{\pi n}\,\sigma(\omega; p\,\pi^0 \to p\,\pi^0)]}{[(P-p)^2+\mu^2]^2} F_1(P)}{\int\limits_{|\vec{t}|>h_3} d_3 \vec{t} \dfrac{(m^2+(pt))}{t_0\,[(t-p)^2+\mu^2]^2} F_1(t)}, \\ c_2 = \dfrac{-1}{4\pi^2\left(\dfrac{g^2}{4\pi}\right)} \dfrac{\int d_4 P \dfrac{[p_0\,\omega\, J_{\pi n}\,\sigma(\omega; p\,\pi^- \to n\,\pi^0)]}{[(P-p)^2+\mu^2]^2} F_2(P)}{\int d_3 \vec{t} \dfrac{(m^2+(pt))}{t_0\,[(t-p)^2+\mu^2]^2} F_2(t)}. \end{cases}$$

F_1 and F_2 being kinematic factors corresponding to cases when finally, we have proton or neutron. We note here that $(p_0\,\omega\, J\,\sigma)$ is an invariant quantity and we can write [3]

$$p_0\,\omega\, J\,\sigma(\omega) = \frac{1}{2}\,[(M^2 - m^2 - \mu^2)^2 - 4\,m^2\,\mu^2]^{1/2}\,\sigma(M^2).$$

In the case when pion-pion scattering at an energy corresponding to our calculations is predominated by a resonance scattering, we can derive, in complete analogy to the Dremin and Černiavski result (we assume the pion-pion compound state to decay isotropically in its own rest frame), the following expressions

$$(8)\quad \begin{cases} c_1 = \dfrac{-1}{2\pi^2\left(\dfrac{g^2}{4\pi}\right)} \dfrac{\int d_4 P \dfrac{[p_0\,\omega\, J_{\pi n}\,\pi(\omega; p\,\pi^0 \to p\,\pi^0)]\,[\omega_0\,\omega\, J_{\pi n}\,\sigma(\omega; \pi^0\,\pi^- \to \pi^0\,\pi^-)]}{[(P-p)^2+\mu^2]^2}}{\int d_3 \vec{t} \dfrac{(m^2+(pt))\,[\omega_0\,\omega J_{\pi n}\,\sigma(\omega; \pi^0\,\pi^- \to \pi^0\,\pi^-)]}{t_0\,[(t-p)^2+\mu^2]^2}}, \\ c_2 = \dfrac{-1}{4\pi^2\left(\dfrac{g^2}{4\pi}\right)} \dfrac{\int d_4 P \dfrac{[p_0\,\omega\, J_{\pi n}\,\sigma(\omega; p\,\pi^- \to n\,\pi^0)]\,[\omega_0\,\omega\, J_{\pi n}\,\sigma(\omega; \pi^-\,\pi^- \to \pi^+\,\pi^-)]}{[(P-p)^2+\mu^2]^2}}{\int d_3 \vec{t} \dfrac{(m^2+(pt))\,[\omega_0\,\omega\, J_{\pi n}\,\sigma(\omega; \pi^-\,\pi^- \to \pi^+\,\pi^-)]}{t_0\,[(t-p)^2+\mu^2]^2}}. \end{cases}$$

Actually, we know nothing about $\sigma_{\pi n}$ but in the first approximation we can try to estimate the effect of such a process by inserting a δ-function instead of $\sigma_{\pi n}$.

The calculation of the ratio $R^{(\exp)}/R$ can be done, but leads to at least one numerical integration. However, in order to perform this calculation, we would be obliged to choose particular experimental conditions.

We shall discuss the estimation of the value of $R^{(\exp)}/R$ and of its energy dependence in a subsequent paper, as well as several other problems connected with properties of the pion-pion interaction.

I should like to express my sincere thanks to Professor Z. Koba for his advice and to Professors W. Królikowski and J. Werle for their kind interest in my work. I am also indebted to M. A. Filipkowski and M. R. Sosnowski for their helpful information on experimental matters.

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Appendix

We shall calculate the kinematic factors F_1 and F_2 only when we take such cases into account, where charged pions are emitted forward (in cms) with momentum larger than r. Such calculations are rather trivial, but we give them here for the sake of completeness. To simplify calculations, we neglect the pion masses (notice that pion energies are large) and assume that the final nucleon (Fig. 1a) or the isobar (Fig. 1b) are emitted backwards, in the direction of motion of the incident proton (the most probable case). Generally speaking, calculations may be performed by the same method, but the results are more laborious to be obtained. Let $\vec{b}$, b_0, M denote momentum, energy and mass of the recoil nucleon (isobar). We have (in cms)

$$F_1 = \int \frac{d_3 \vec{k}_1}{k_1} \int \frac{d_3 \vec{k}_2}{k_2} \delta(\vec{b} + \vec{k}_1 + \vec{k}_2)\, \delta(p_0 + p - b_0 - k_1 - k_2) \times$$

$$\times \vartheta(k_1 - r)\, \vartheta(-\vec{k}_1 \vec{b}) = 2\pi \int_{-1}^{1} d\cos\Theta \int_0^{k_{\max}} dk \frac{k}{\sqrt{b^2 + k^2 + 2bk\cos\Theta}} \times$$

$$\times \delta(p_0 + p - b_0 - k - \sqrt{b^2 + k^2 + 2bk\cos\Theta})\, \vartheta(k - r)\, \vartheta(-kb\cos\Theta).$$

$k_{\max}$ is given by the formula $k_{\max} = \frac{(p_0 + p - b_0 + b)}{2}$ and $\vartheta(x)$ denotes the step function

$$\vartheta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

We perform the substitution

$$z = k + \sqrt{b^2 + k^2 + 2bk\cos\Theta}.$$

Thus,

$$F_1 = \frac{2\pi}{b}\int_0^{k_{\max}} dk \int_{k}^{k+\sqrt{(b+k)^2}} dz\,\delta(p_0+p-b_0-z)\,\vartheta(k-r)\,\vartheta(b^2+2zk-z^2) =$$

$$= \frac{2\pi}{b}\int_0^{k_{\max}} dk\,\vartheta(k-r)\,\vartheta(b^2+2k(p_0+p-b_0)-(p_0+p-b_0)^2)\times$$

$$\times\vartheta(p_0+p-b_0-k-\sqrt{(b-k)^2}) =$$

$$= \begin{cases} \frac{2\pi}{b}\,(k^{(1)}-k^{(2)}) & p+p_0 > b+b_0 \\ 0 & p+p_0 < b+b_0, \end{cases}$$

where

$$k^{(1)} = k_{\max}$$

$$k^{(2)} = \max\begin{cases} v \\ \dfrac{(p_0+p-b_0)^2-b^2}{2(p_0+p-b_0)} \end{cases}.$$

We see that the momentum transfer is limited from below, since $p_0+p>b_0+b$ gives approximately *) $\Delta p\;\;0\,|\Delta p|>\dfrac{m\,\Delta m}{p+p_0}$ where $\Delta p = b-p$, $\Delta m = M-m$. If we take into account pion mass we get $|\Delta p|>\dfrac{2p\,\mu^2}{3p_0(p_0+\omega_0)}+\dfrac{pm\,\Delta m}{\omega_0(p+\omega_0)}$. In the same manner we obtain for F_2:

$$F^2 = \int \frac{d_3\vec{k}_1}{k_1}\int \frac{d_3\vec{k}_2}{k_2}\,\delta(\vec{b}+\vec{k}_1+\vec{k}_2)\,\delta(p+p_0-b_0-k_1-k_2)\times$$

$$\times\vartheta(k_1-r)\,\vartheta(k_2-r)\,\vartheta(-\vec{k}_1\cdot\vec{b})\,\vartheta(-\vec{k}_2\cdot\vec{b}) =$$

$$= \begin{cases} \frac{2\pi}{b}\,(k^{(3)}-k^{(4)}) & p_0+p>b_0+b \\ 0 & p_0+p<b_0+b, \end{cases}$$

where

$$k^{(3)} = \min\begin{cases} k_{\max} \\ (p_0+p-b_0-r), \end{cases}$$

$$k^{(4)} = \max\begin{cases} v \\ \dfrac{(p_0+p-b_0)^2-b^2}{2(p_0+p-b_0)} \end{cases}.$$

*) When we neglect the higher order terms in Δm

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High-Energy Pion-Nucleon Collisions and Isotopic Properties of the Pion-Pion Interaction

by

A. KRZYWICKI

Presented by L. INFELD on July 14, 1960

In this communication we use the same notations and assumptions as in the preceding paper [4].

In the previous paper we expressed $R^{(\mathrm{exp})}/R$ in terms of known observable quantities. We found that

$$\frac{R^{(\mathrm{exp})}}{R} = \frac{1+C_1}{1+C_2},$$

where C_1 and C_2 are given by the formulae (7) in [4]. In sec. **1** we shall roughly estimate the magnitude and energy dependence of $R^{(\mathrm{exp})}/R$. In sec. **2** the dependence of R on isotropic properties of the pion-pion interaction is briefly discussed. In sec. **3** an experiment is proposed and in sec. **4** some comments on the problem of cloud-cloud interaction in high-energy nucleon-nucleon collisions are added.

1. We shall not calculate here exactly C_1 and C_2. We think, however, that a rough estimation of the order of magnitude and general behaviour of C_1 and C_2 is necessary.

Let us observe that in order to estimate C_1 and C_2 it is sufficient to restrict the domain of integration to small Θ and $|\vec{t}-\vec{p}|(|\vec{P}-\vec{p}|)$. We assume also that only a 3/2, 3/2 isobar is produced. It was shown ([4], Appendix) that in the case, when the isobar is produced, momentum transfer is considerably limited from below. Taking into account all these conditions we get: *)

$$(1)\quad \int d_4P \frac{(p_0\omega J_{\pi n}\sigma(\omega))}{[(P-p)^2+\mu^2]^2} F(P) \sim \frac{\pi\Theta^2_{\max} p^2 F(\widetilde{P})}{\mu^4 p_0} \times$$

$$\times \int \frac{1}{2} M[(M^2-m^2-\mu^2)^2-4m^2\mu^2]^{1/2}\sigma(M^2)\left[(\Delta p)_{\max} - \frac{m(M-m)}{p_0+\omega_0}\right],$$

*) $(\Delta p)_{\max}$ denotes there that the absolute value of the maximal momentum transfer occurring in a peripheral collision, is of the order of μ and plays the role of a parameter.

and

$$(1') \qquad \int d_3\vec{t}\,\frac{(m^2+(pt))}{t_0[(t-p)^2+\mu^2]^2}F(t)\sim -\frac{\pi\Theta_{\max}^2 p^2 m^2 F(\tilde{t})(\Delta p)_{\max}^3}{6\mu^4 P_0^3}.$$

Thus,

$$(2) \qquad \frac{9}{8}c_1\sim\frac{9}{2}c_2\sim\frac{p_0^2}{\left(\frac{g^2}{4\pi}\right)(\Delta p)_{\max}^3}\left[d_1(\Delta p)_{\max}-\frac{d_2 m^2}{p_0+\omega_0}\right]$$

where

$$(3) \qquad \begin{cases} d_1=\dfrac{3}{4\pi^2 m^2}\displaystyle\int M[(M^2-m^2-\mu^2)^2-4m^2\mu^2]^{1/2}\sigma_{3/2}(M^2)\,dM, \\ d_2=\dfrac{3}{4\pi^2 m^2}\displaystyle\int M(M-m)[(M^2-m^2-\mu^2)^2-4m^2\mu^2]^{1/2}\sigma_{3/2}(M^2)\,dM \end{cases}$$

Rough numerical calculation gives

$$(4) \qquad \begin{aligned} d_1 &\sim 3.7 \\ d_2 &\sim \frac{\mu}{m}\times 9.3. \end{aligned}$$

We see that $R^{(\exp)}/R$ increases with energy and is of the order of unity. We have

$$(5) \qquad 1\lesssim\frac{R^{(\exp)}}{R}\lesssim 4.$$

At the energy corresponding to Walker's experiment [1], 4.5 GeV, C_1 is of the order of 10, so that $R^{(\exp)}/R$ is not far from 4.

2. As an illustration let us discuss the dependence of R on the isotopic properties of the pion-pion interaction. We have (B denotes the physical pion-nucleon vertex):

$$(6) \qquad R=\frac{\int_{\vec{t}>h_3} d_3\vec{t}\,\dfrac{|A_1|^2|B_1|^2}{t_0[(t-p)^2+\mu^2]^2}F_1(t)}{\int d_3\vec{t}\,\dfrac{|A_2|^2|B_2|^2}{t_0[(t-p)^2+\mu^2]^2}F_2(t)},$$

where A_1, B_1 and A_2, B_2 correspond to cases when we finally have a proton or neutron. Let us assume sufficiently weak energy and angle dependence of A. Then R is essentially determined by the value of the quantity

$$\frac{|A(\pi^0\pi^0\to\pi^0\pi^0)|^2}{2|A(\pi^+\pi^-\to\pi^+\pi^-)|^2} \qquad \left(\frac{|B_1|^2}{|B_2|^2}=\frac{1}{2}\right)$$

since in the first approximation we can neglect the influence of different kinematic factors in each integral. These factors select the most probable events. Using the simplest pion-pion interaction $\lambda\varphi^4$ and calculating the pion-pion vertex by the perturbation method, one gets $R\sim 1/8$ [2].

The amplitudes A_1 and A_2 can be expressed in terms of three amplitudes $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ corresponding to three possible total isospins of the two-pion system

$$A_1 = \frac{1}{2} A^{(1)} + \frac{1}{2} A^{(2)},$$

$$A_2 = \frac{1}{3} A^{(0)} + \frac{1}{2} A^{(1)} + \frac{1}{6} A^{(2)}.$$

As was already noted, we can write approximately

$$R \sim \frac{A(\pi^0\pi^- \to \pi^0\pi^-)|^2}{2|A(\pi^-\pi^- \to \pi^+\pi^-)|^2}. \tag{7}$$

We neglect the dependence of $A^{(i)}$ on variables other than the isotopic spin. We shall consider seven extreme cases (phase differences between $A^{(i)}$'s are not taken into account):

$$|A^{(0)}| \gg |A^{(1)}|, |A^{(2)}|$$

$$|A^{(1)}| \gg |A^{(0)}|, |A^{(2)}| \qquad |A^{(2)}| \gg |A^{(0)}|, |A^{(1)}| \qquad |A^{(0)} \sim A^{(1)}| \gg |A^{(2)}|,$$

$$|A^{(0)} \sim A^{(2)}| \gg |A^{(1)}| \qquad |A^{(1)}| \sim |A^{(2)}| \gg |A^{(0)}| \qquad |A^{(0)} \sim A^{(1)}| \sim |A^{(2)}|.$$

Table I gives R (calculated by means of the formula (7)) in these extreme cases. The first line of the Table shows isotopic spins which

TABLE I

T	(0)	(1)	(2)	(0,1)	(0,2)	(1,2)	(0.1.2)
R	0	0,5	4,5	0,18	0,5	1,125	0,5

characterize "large" $A^{(i)}$'s, e.g. to the case $|A^{(1)} \sim A^{(2)}| \gg A^{(0)}$ corresponds the symbol (1,2) in the first line.

3. The examination of the process $\pi^- + p \to \pi^- + \pi^+ + \pi^- + p$ can serve as a possible cross-check of our previous considerations. We assume that the mechanism of a peripheral process of this type is the pion--pion interaction leading to isobar formation. Then a particle which was produced by the decaying isobar and flies backwards in cms appears as a relatively slow particle in the lab. system. Concider a bubble chamber experiment. We take into account only stars, where at least two particles are very energetic and one relatively slow (in lab. system) and where the momentum is balanced. Let us select stars in such a manner that the slow particle is identified as a pion. Since we can measure charges of fast particles, the conservation of charge gives us that of the slow pion. The slow pion was emitted in cms backwards and with considerable momentum, so that we can be almost sure that it resulted from the decay of the isobar. We can measure the relative frequency of occurrence of

stars with slow pion being positive or negative (we denote this quantity as R_1). Using the approximations made before, we get (we assume that only 3/2, 3/2 isobar is produced):

$$R_1 \sim \frac{9|A(\pi^-\pi^- \to \pi^-\pi^-)|^2}{|A(\pi^-\pi^+ \to \pi^-\pi^+)|^2}.$$

Table II shows R_1 in seven extreme cases (we neglect the dependence $A^{(i)}$ on other variables than the isotopic spin)

TABLE II

T	(0)	(1)	(2)	(0,1)	(0.2)	(1,2)	(0,1,2)
R_1	0	0	324	0	36	20,25	9

As it should be expected, the behaviour of R_1 depends very strongly on the magnitude of pion-pion scattering in $T=2$ state. When this scattering is much smaller than in other isotopic states we observe mostly negative pions emitted backwards (in cms). If it is comparable to the scattering in other states we should observe mostly positive pions emitted backwards in cms. Using the same method of estimation as in sec. **1** and neglecting the energy dependence of A and F we get for the cross-section of the peripheral process $\pi^- + p \to \pi^- + \pi^- + \pi^+ + p$ in the case when π^- is emitted backwards:

$$\sigma^{\text{periph}}_{\pi \text{ back}}(\pi^- p \to 2\pi^-\pi^+ + p) = \text{const} \frac{p^2}{p_0^2 \omega_0}\left[d_1(\Delta p)_{\max} - \frac{d_2 m^2}{p_0 + \omega_0}\right].$$

We find that $\sigma^{\text{periph}}_{\pi-\text{back}}(\pi p \to 2\pi^-\pi^+ p)$ corresponding to the pion lab. energy 4.5 GeV is of the order of, or is somewhat smaller than, $\sigma_{\text{periph}}(\pi^- p \to \to \pi^-\pi^+ n)$, that is probably of an order of 1 mb.

4. The problem of the role of the cloud-cloud interaction in high-energy peripheral nucleon-nucleon collisions is closely connected with the isotopic properties of the pion-pion interaction. In a previous paper [3] we have shown, using the simplest interaction $\lambda\varphi^4$, that an asymmetry between final protons and neutrons in inelastic p-n collisions just opposite to the experimentally observed one is obtained. We shall here make some additional comments on this point. Let us assume first that two virtual pions of two clouds corresponding to colliding nucleons, interact in an S-state.

The asymmetry parameter R_2 approximately equals

$$R_2 = \frac{1}{4}\frac{|A(\pi^0\pi^0 \to \pi^0\pi^0)|^2 + |A(\pi^0\pi^0 \to \pi^+\pi^-)|^2}{|A(\pi^+\pi^- \to \pi^+\pi^-)|^2 + |A(\pi^+\pi^- \to \pi^0\pi^0)|^2}.$$

Table III shows the dependence of R_2 on isotopic properties of the pion-pion interaction (experimentally $R_2 > 1$).

TABLE III

T	(0)	(1)	(2)	(0,1)	(0,2)	(1,2)	(0,1,2)
R_2	1/4	—	1	—	1	—	—

Note that here we speak of the pion-pion scattering at energies much lower than before; if we consider pion-pion scattering, then nucleon--nucleon collision at the energy E_{lab} corresponds to the pion-nucleon scattering at the energy $\sim \mu_\pi m \cdot E_{\text{lab}}$. If we assume the predominance of P-wave pion-pion scattering, $R_2 < 1$ since, when two π^0's are present in the initial or final state, P-wave scattering does not occur due to the Pauli principle.

Finally, I avail myself of the opportunity to express my sincere thanks to Professor Z. Koba for his advice as well as to Professor W. Królikowski and J. Werle for kind interest in my work. I am indebted also to M. A. Filipkowski and M. R. Sosnowski for their helpful information on experimental matters.

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GEOPHYSICS

On the Superadiabatic Lapse Rate in the Upper Air

by

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Presented by H. NIEWODNICZAŃSKI on April 27, 1960

Introduction

The superadiabatic lapse rate of temperature, i.e. the fall of temperature with altitude, greater than $\Gamma = Ag/c_p \approx 1^\circ\text{C}/100\text{ m}$ *) is but rarely observed in the upper air, as it is connected with an unstable thermodynamical equilibrium of the atmosphere and, therefore, disappears very quickly, if no mechanism maintains its existence. The most common mechanism of this type is the differential heating by sun radiation of two layers with different absorption properties, such as the ground surface and air, clouds and clear air, etc. Sometimes, the superadiabatic lapse rate can be produced on the frontal surfaces or by the advection of cold in the upper air. It can also be found within or near the boundaries of convective motions and clouds, where air masses of different temperature remain next to each other in unstable conditions.

In all investigations of the superadiabatic lapse rate it must be remembered, that the lapse rate obtained in the radiosounding by means of a free balloon, called the "balloon lapse rate" coincides with the true lapse rate only, if the temperature distribution during the sounding depends solely on the altitude.

It is also necessary to eliminate the instrumental or semi-instrumental errors such as sun radiation errors, instrumental inertia, etc.

Although cases of superadiabatic lapse rate are recorded from time to time by various aerological stations, theoretical works concerning the reasons and mechanism of this phenomenon are extremely scarse. The author of the present paper knows of only one such paper [1] **) in which

*) where A denotes thermal equivalent of work, g — gravitational acceleration, c_p — specific heat of air under constant pressure, Γ — the so-called dry adiabatic lapse rate.

**) The author became acquainted with the paper [3] concerning some cases of SLR on the interface between clouds and dry air, when the present paper was already submitted for publication.

W. I. Godson proves, that when exclusively dry adiabatic processes take place, the instant "balloon lapse rate" cannot be superadiabatic if the true rate is not so, though the magnitudes of both these rates may be different. Godson's considerations can be easily extended to the case of the mean lapse rate (i.e. measured between two finite-distant points by a radiosonde with finite velocity) in this sense, that the mean "balloon lapse rate" cannot be superadiabatic, if there is no instant local superadiabatic lapse rate at some points of the balloon trajectory, in the case when exclusively dry adiabatic processes are possible.

Godson supposes that the great majority of cases of superadiabatic lapse rate observed above the layer of convection is connected with the non adiabatic processes in stably stratified air, caused by the evaporation of rain. Godson suggests advection as another reason, although this seems to be rather rare.

It may be proved that in the case of radiosondes with non-continuous registration the superadiabatic lapse rate may be observed in exceptional situations as an effect of dynamical pressure changes.

The author has observed many cases of the superadiabatic lapse rate at altitudes ranging from 10 to 17 km, [2]. This cannot be explained satisfactorily by instrumental errors or by any of the physical reasons mentioned above. In what follows, a new hypothesis will be advanced in which the author suggests that the superadiabatic lapse rate in the upper air may be produced by forced, dynamical turbulence in stably stratified air.

Basic formulae

Let T denote the temperature in Kelvin degrees, p — pressure in millibars, Θ — potential temperature, R — the gas constant for dry air. (Other symbols as before).

Let us introduce the local co-ordinate system x, y, z with the z axis directed vertically upwards. The temperature lapse rate is then given by $-\partial T/\partial z$. The following relations hold:

$$\Theta = T\left(\frac{1000}{p}\right)^{\frac{AR}{cp}}, \tag{1}$$

(Θ is an invariant of dry-adiabatic processes);

$$\frac{\partial \Theta}{\partial z} = \frac{\Theta}{T}\left(\Gamma + \frac{\partial T}{\partial z}\right) \tag{2}$$

(the superadiabatic lapse rate corresponds to negative values of $\partial\Theta/\partial z$).

Let us describe the air motion by means of Lagrange variables, i.e. by a system of functions

$$x = x(x_0, y_0, z_0, t), \quad y = y(x_0, y_0, z_0, t), \quad z = z(x_0, y_0, z_0, t), \tag{3}$$

describing the trajectory of a particle of air, the position of which at the initial moment t_0 was given by (x_0, y_0, z_0). We assume that these functions are reversible in respect to x_0, y_0, z_0. Interpreting the notion of an air particle as a sufficiently small but finite volume, this last assumption means that the individuality of the particle is preserved throughout the time of motion, thus the effects of mixing may be neglected.

Let index $_0$ refer to the situation at the initial moment t_0. Assuming the motions to be adiabatic, we have

$$\Theta(x, y, z, t) = \Theta_0(x_0, y_0, z_0). \tag{4}$$

Assuming that Θ_0 depends only on z_0, we obtain:

$$\frac{\partial \Theta}{\partial z} = \frac{\partial \Theta_0}{\partial z_0} \frac{\partial z_0}{\partial z}, \tag{5}$$

this means that in the initially stable $(\partial\Theta_0/\partial z_0 > 0)$ air a superadiabatic lapse rate can be produced by motion if $\partial z_0/\partial z < 0$. Physically this condition corresponds to the change of order of the particles on the vertical axis. Example 1 illustrates this more clearly.

From (2) and (5) we obtain the following expression for the temperature lapse rate at moment t adopting the initial lapse rate $-\partial T_0/\partial z_0$:

$$\frac{\partial T}{\partial z} = \Gamma - \frac{T}{\Theta}\frac{\partial \Theta}{\partial z} = \Gamma - \frac{T}{\Theta}\frac{\partial \Theta_0}{\partial z_0}\frac{\partial z_0}{\partial z} = \tag{6}$$

$$= \Gamma - \frac{T}{\Theta}\frac{\Theta_0}{T_0}\left(\Gamma + \frac{\partial T_0}{\partial z_0}\right)\frac{\partial z_0}{\partial z} = \Gamma - \frac{T}{T_0}\left(\Gamma + \frac{\partial T_0}{\partial z_0}\right)\frac{\partial z_0}{\partial z}.$$

This formula contains the well known laws of change of the temperature lapse rate in upward and downward motions; it suffices to remark that $(T/T_0)(\partial z_0/\partial z) \to 0$ for ascending motions and $(T/T_0)(\partial z_0/\partial z) \to \infty$ for descending ones.

Simplifications

The error in aerological data concerning the temperature surely exceeds 0.2°C. Therefore, assuming that the pressure and the altitude are given with ideal accuracy, we obtain for the error of the lapse rate (calculated for instance by means of a thermodynamical diagram) a value of at least $\pm 0.05\ \Gamma$, provided that the distance between the characteristic levels does not exceed 1 km (this distance in a great majority of observed cases may be accepted as the maximum thickness of a layer with superadiabatic lapse rate).

The above remark allows to introduce all simplifications of formulas, if only final error for the lapse rate introduced by these simplifications is smaller than $0.05\ \Gamma$. For instance, let us estimate: $\Gamma > -\partial T_0/\partial z_0 \geqslant 0.5$, $\partial z_0/\partial z < 2$ which is valid for many real situations. Assuming that the deepness of the investigated motions does not exceed 1 km and that

$T_0 > 210°$ K, we can put $T/T_0 = 1 \pm \varepsilon$, $\varepsilon \leqslant 0{,}005$. Thus (6) can be replaced with sufficient accuracy by:

$$-\frac{\partial T}{\partial z} = \Gamma - \left(\Gamma + \frac{\partial T_0}{\partial z}\right)\frac{\partial z_0}{\partial z}. \tag{7}$$

The latter formula is particularly convenient in application. It follows from (2) and (7) that $\partial\Theta/\partial z$ and $\partial T/\partial z$ depend only on the initial situation and on the kinematical properties of motions. For the small-scale (1000 m) motions it may be more convenient to base the investigations not on the hydrodynamical equations but on the following assumptions:

1. Functions x, y, z are known.
2. Pressure distribution $p(x, y, z, t)$ is known (generally as the hydrostatical pressure depending only on the altitude z).
3. Initial temperature distribution $T_0(x_0, y_0, z_0)$ is known.
4. Motions are adiabatic (i.e. $Tp^{\frac{AR}{c_p}} = \text{const}$ for an individual particle).

Generally x, y, z, p and T will not fulfil the exact hydrodynamical equations, but they will only form a satisfactory thermohydrodynamical model for investigations of the lapse rate changes. The use of such a model is additionally substantiated by the fact, that the local pressure differences connected with the small-scale motions are generally very small and not measurable by means of commonly used aerological equipment. The pressure measured by the radiosonde even in a region of local disturbances can be assumed to be equal to the hydrostatic pressure characteristic of the undisturbed environment. Of course, not for every system of functions x, y, z, p and T will there exist a true motion for which this system will form a sufficiently exact model, but mathematical criterions as to whether such a system has physical sense, exceed the subject of the present paper. Specially interesting are movements which can be approximated with sufficient exactness by the motions of a rigid body or an incompressible fluid, for the mathematical form of the functions x, y, z may be then comparatively simple.

Examples

Example 1. Let us assume that at the initial moment $T_0 = T^* - \beta z_0$, $\beta < 1$; pressure $p = p^* \exp\left(\int_0^z g dz_0 / RT_0\right)$ remains constant during the motion. Let the motion have such properties that at the moment t there exists such a pair of points (x, y, z_1) and (x, y, z_2) that $z_1 > z_2$ and $z_0(x, y, z_1, t) < z_0(x, y, z_2, t)$. Of course, $\partial\Theta_0/\partial z_0 > 0$ and hence $\Theta_0[z_0(x, y, z_1, t)] < \Theta_0[z_0(x, y, z_2, t)]$; but from the invariancy of Θ it follows that at the moment t

$$\Theta(x, y, z_1, t) < \Theta(x, y, z_2, t)$$

and hence

$$\frac{\Theta(x, y, z_1, t) - \Theta(x, y, z_2, t)}{z_1 - z_2} < 0 .$$

Applying the Lagrange mean-value-theorem to the last inequality, it may be easily seen that there must exist such a point, at which $\partial\Theta/\partial z < 0$, i.e. at which the lapse rate is superadiabatic. Properties of the motion as required above are characteristic for different eddy motions. The simplest motion of this kind will be discussed in the following:

Example 2. Let us consider a rigid eddy motion, uniform with regard to the y-axis, given by the equations

$$\begin{cases} x = x_0 \cos\omega t - z_0 \sin\omega t & x_0 = x\cos\omega t + z\sin\omega t \\ & \text{i. e.} \\ z = x_0 \sin\omega t + z_0 \cos\omega t & z_0 = -x\sin\omega t + z\cos\omega t \end{cases} \tag{8}$$

(corresponding to a rigid rotation around the y axis with angular velocity ω).

Applying the formula (7) we obtain

$$-\frac{\partial T}{\partial z} = \Gamma - (\Gamma - \beta)\cos\omega t,$$

i.e. the lapse rates oscillate around the adiabatic lapse rate with an amplitude $\Gamma - \beta$.

It should be noted that such a rigid motion forms a satisfactory approximation of some true eddy motion of sufficiently small diameter. For instance, at the level of 200 mb this model approximates some true eddy of diameter ca. 500 m with an accuracy of ca. 10 m which is sufficient for our purposes. The limit for the lapse rate available in such motion $2\Gamma - \beta$ is characteristic for all motions approximable by rigid motions, for in this case $\partial z_0/\partial z$ is an element of the orthonormal matrix and hence $\partial z_0/\partial z \leq 1$.

Conclusions

Observing wave clouds, dust in turbulent motions, etc. one can see that such motions often exhibit the properties of Example 1 or even they can be partially approximated by motions similar to Example 2. It is very probable that dynamic clear air turbulence also has these properties; in such cases the superadiabatic lapse rate may be perhaps observed in the turbulent layer. Of course, once the superadiabatic lapse rate occurs, the mixing by thermal convection will play an important role and the above theory does not hold any more. The mixing, both convectional and dynamic, cannot also be neglected in this turbulent layers, therefore in these cases observations of the superadiabatic lapse rate by means of the common radiosonde may be difficult. If, however, the diameter of the

eddy is large — perhaps greater than 500 m — the life of the superadiabatic lapse rate may be as long as a few minutes, it can therefore be detected. Such great eddies belong probably to rare phenomena, and this may be the reason why the superadiabatic lapse rate in the upper air is only occasionally observed.

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GEOPHYSICS

On the Observations of the Superadiabatic Lapse Rate in the Vietnamese-Polish IGY Observatory Cha-Pa

by

K. HAMAN

Presented by H. NIEWODNICZAŃSKI on April 27, 1960

Introduction

The Vietnamese-Polish IGY Observatory Cha-Pa (22°21′N, 103°50′E, 1578 m a.s.l.) was established in July 1957 as a result of Vietnamese-Polish collaboration in connection with the International Geophysical Year (IGY). It lies in the mountains, about 10 km from the highest peak of Indochina (Phan-Si-Pan — 3142 m a.s.l.). The aerological station operating there among other geophysical disciplines, had in its program four radiosoundings daily at 00, 06, 12, 18 GMT (07, 13, 18, 01 LT) on all RWD, SWI and WMI, starting from August 1957 and, moreover, one radiosounding at 00 GMT every day, starting from January 1, 1958. The radiosoundings were completed by optical pilotages, if only the visibility was satisfactory. For sounding the Vaisala Sounding System was used. During the IGY more than 700 radiosoundings were made at Cha-Pa. Among them in more than 200 soundings characteristic disturbances of the pressure-temperature curve have been detected in the troposphere, which in Cha-Pa reaches the level of ca 100 mb. Each such disturbances consisted of a layer "A", with inversion or isotherm, and of layer "B", with the superadiabatic lapse rate (abbr. SLR). The materials concerning this phenomenon have already been partially elaborated and the present paper contains preliminary results of these investigations.

Preliminary results of empirical investigations

In about 70% *) of the observed disturbances layer "A" lies below layer "B" and mostly is not thicker than the latter. Generally, the disturbances appear singularly, but about 30% of soundings with this phenomenon present demonstates series of 3—5 or more disturbances one

*) All statistical data in this report have only an orientational character. They are based on the results of soundings made up to June 30, 1958, i.e. on relatively small material.

above the other. The thickness of layer "B" varies between 100—700 m (exceptionally more than 1000 m), generally between 300—500 m. The thickness of layer "A" is of the same order, or even somewhat smaller. The deviation of temperature from the smoothed pressure-temperature line in the extremal points of disturbances, varies from 1° to 3°C. The smoothed lapse rate of temperature in the layer with disturbances varies from 0.6°C/100 m to 0.9°C/100 m, the lapse rate of layer "A" is generally about — 1°C/100 m, and the lapse rate of layer "B" varies from 1.1°/100 m to 1.7°C/100 m.

Examples of disturbances are shown in Figs. 1 and 2.

Fig. 1. Vaisala Sounding System record of a typical series of disturbances observed in Cha-Pa. Sounding No. 385, 15.5.1958, 7.00 a. m. LT.

The seasonal distribution of disturbances is difficult to determine on the basis of the available material, but it seems that the late winter and spring are nearly free from the disturbances, while the greatest frequency is observed between May and September.

The diurnal distribution of the frequency of disturbances on different altitudes is given in the following Table:

TABLE

Percentual distribution of soundings with disturbances according to the level of disturbances and the hour of sounding

Daytime GMT / Altitude	Day (00,06 GMT)	Night (12,18 GMT)	Total
below 300 mb	(1) 15%	(2) 15%	30%
above 300 mb	(3) 60%	(4) 10%	70%
Total	75%	25%	100%

Each sounding in which the disturbances were detected, was included into the group "below 300 mb" or "above 300 mb" depending on the level at which the disturbances had appeared or maybe into both. 100% corresponds to the sum of soundings in both groups, which may be greater than the total amount of soundings with disturbances. Numbers in brackets denote the number of the entry to facilitate references in the further text.

Let us remark, that the series appear generally above the 300 m level and nearly always during daytime. The richest in disturbances is the sounding at 06 GMT (13 LT), the poorest — at 12 GMT (19 LT).

Disturbances (1), (2) and (4) (see Table) are generally weak, i.e. they exhibit small thicknesses, small deviations and only slight SLR.

Some efforts were made to find a correlation between the appearance of disturbances and clouds, but without satisfactory results, because the synoptical data on clouds do not always inform about the clouds crossed by the balloon. Moreover, low orographical clouds have very often rendered the observations of higher clouds impossible. It seems that a considerable part of the disturbances have been observed in clear air.

In August and September 1959 special investigations of disturbances were made. They were based on soundings made with the use of two radiosondes launched together on the same balloon or at short intervals one after the other. Unfortunately, contrary to the situation in previous years, the disturbances in sufficiently strong form have appeared only twice. Although they were detected by a single radiosonde, a couple of radiosondes on the same balloon launched about one hour later did not detect anything. Probably the scope and lifetime of this phenomenon are very restricted.

It is clear, that such disturbances as discussed above, may not be recorded by radiosondes with discontinuous registration of data, because radiosondes of this type "cut off" the characteristic points. Therefore, data from other stations are only comparable, if the latter are equipped with continuous registration radiosondes. Chinese aerological stations working in South China with the Vaisala Sounding System were asked, whether they had detected phenomena similar to the discussed disturbances. These stations answered that at any rate no stronger disturbances had been observed there. This suggests that the disturbances are a local effect of Cha-Pa.

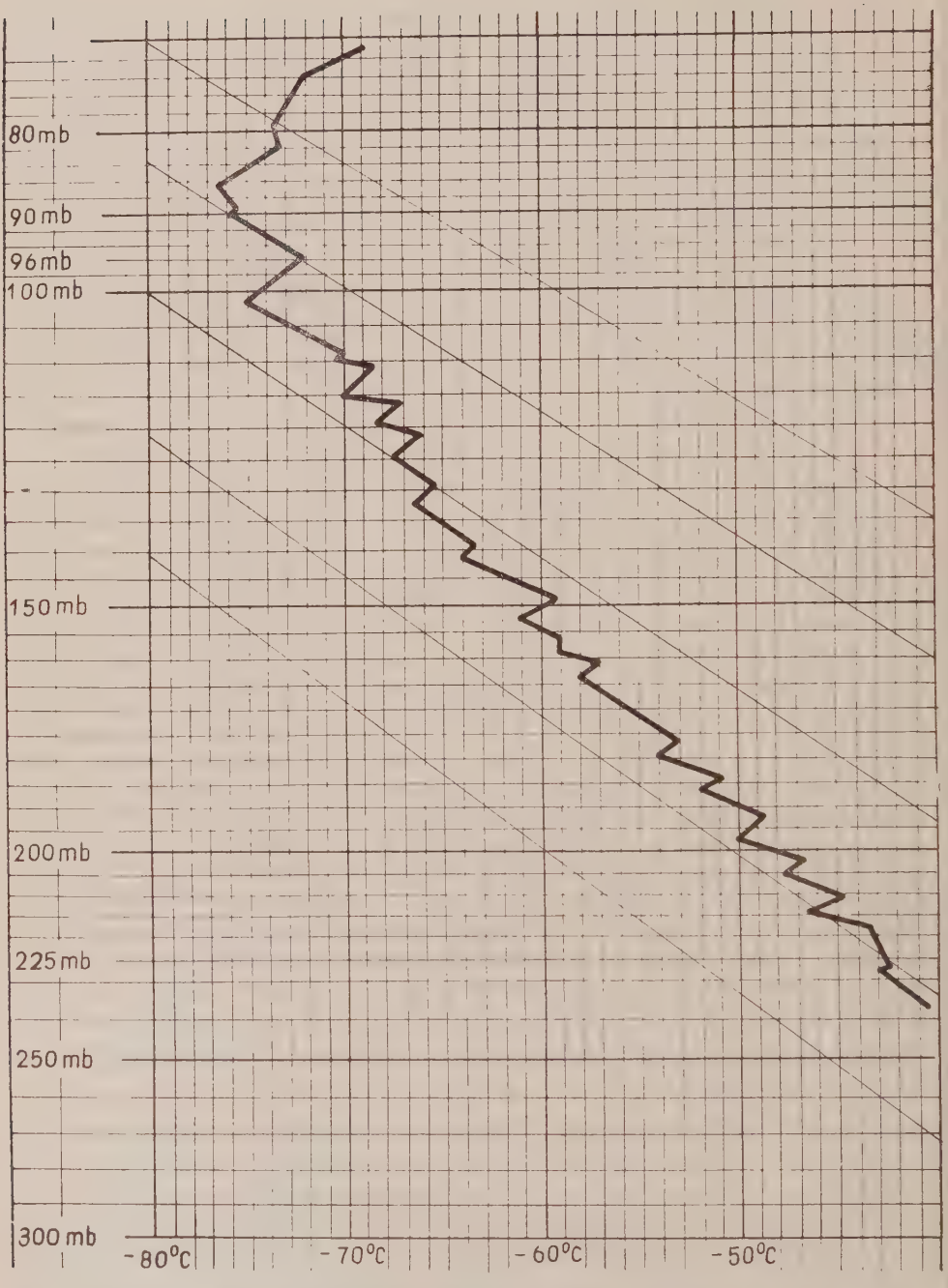

Fig. 2. Fragment of Stueve diagram of souding containing a typical series of disturbances. Sounding Cha-Pa No. 385, 15.5.1958 7.00 a.m. LT. Diagonal lines — dry adiabats

Attempt of explanation

It is visible from the Table, that the frequency of appearance of disturbances in positions (1), (2), (4) is approximately equal, while in position (3) — considerably higher. This suggest the existence of two groups of disturbances. The first one, including the disturbances of the positions (1), (2), (4) and probably a part of the disturbances from position (3), does not depend on daytime or on altitude. Disturbances of this group were generally weak and single. These disturbances were probably caused by different accidental factors, such as clouds, dust layers, dynamical inversions, water condensation and evaporation, etc. Double radiosoundings made in 1959 have shown, that in many cases, the reason of such disturbances may lie in purely instrumental effects.

The second group are disturbances, occurring in the daytime at great altitudes between the 300 mb level and the tropopause. These disturbances were often strong and often appeared in series.

Explanation of the origin of this group is more difficult.

A priori the possible causes may be classified as follows: a) a purely instrumental effect without true SLR in the atmosphere, b) instrumental effect caused by an external physical reason, however, without true SLR in the atmosphere, c) true SLR in the atmosphere.

The results of investigations carried on so far show that the first two groups of reasons are not very probable, however any definite conclusion is not yet possible.

A purely instrumental effect would not depend on daytime and place. In group b) the water phase changes and sun radiation may be taken under consideration. Water phase changes can be excluded, for at the 200 mb level and at —50°C their thermic effect is practically negligeable. The sun radiation effect may change, due to variations of the relative balloon velocity, changes of the angle between the thermometer shield and the sun (as result of oscillations of the radiosonde under the balloon) and due to the shadow of clouds. In the two first cases the period of temperature oscillations would be much shorter than the observed one. In the last case, several observations show that the shadow effects of Cirrus clouds (Ci, Cc, Cs — only possible at so great altitudes) on the radiation error, with the exception of certain "inc gen" forms, are generally imperceptible. Moreover, in this case the shape of the majority of disturbances on the pressure-temperature curve would be difficult to explain, especially in the case of series of disturbances. The only positive feature of the sun radiation hypothesis is that it explains the diurnal distribution of the disturbances. In group c) the character of disturbances excludes the advection processes as the reason of the SLR. The boundary effects of convectional clouds and vertical motions must be also excluded in this part of disturbances, which occurs above the ceiling of the thermic convection. Therefore, as the only possible cause there remains

clear air turbulence. The possibility of appearance of the SLR in sufficiently large eddies with horizontal axes has already been shown in [1]. It may be, that some special influences of orography lead to the formation of clear air turbulence with eddies of sufficiently large dimensions even on great altitudes. This hypothesis explains the formation of the SLR — even in completely clear air, as a local effect at Cha-Pa. The diurnal distribution of the disturbances may be explained by the supposition, that the formation of high turbulent motion depends essentially not only on the dynamic effects of the mountain ridges (these effects in the region of Cha-Pa are probably small, due to generally weak winds prevailing in this region), but also on the thermic convection in the lower troposphere. Such a supposition referring for example to some kinds of mountain waves is not new.

This preliminary hypothesis contains, of course, many unclear points; some of them will probably be explained after a deeper analysis of the complete IGY aerological data from Cha-Pa.

Research on these problems is being continued.

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